Biquaternions

Hamilton defined the quaternion basis i, j, k like

$$i^2 = j^2 = k^2 = ijk = -1$$

and quaternions to be of the form $q = q_0 + q_1 i + q_2 j + q_3 k$. If we let each q_i be a complex number of the form $a + bh$ with $h^2 = -1$ and commuting with i, j, k, we arrive at the biquaternions. As it turns out the biquaternions are exactly equivalent to the geometric algebra $\mathrm{Cl}_{3,0}(\mathbb{R})$.

$\mathrm{Cl}_{3,0}(\mathbb{R})$ is generated by the basis vectors e_1, e_2, e_3 with $e_i^2 = 1$ and $e_i e_j = -e_j e_i$ for $i \neq j$. We will write e_{ij} for $e_i e_j$ &c.

We find the following multiplication rules:

$$\begin{aligned}
e_{12}^2 &= e_{1212} = -e_{1122} = -1 \\
e_{23}^2 &= e_{2323} = -e_{2233} = -1 \\
e_{31}^2 &= e_{3131} = -e_{3131} = -1 \\
e_{12}e_{23} &= e_{13} = -e_{31} \\
e_{23}e_{31} &= e_{21} = -e_{12} \\
e_{31}e_{12} &= e_{32} = -e_{23} \\
e_{123}^2 &= e_{123123} = e_{112323} = -e_{112233} = -1 \\
e_1 e_{123} &= e_{123}e_1 = e_{23} \\
e_2 e_{123} &= e_{123}e_2 = e_{31} \\
e_3 e_{123} &= e_{123}e_3 = e_{12} \\
e_{23}e_{123} &= e_{123}e_{23} = -e_1 \\
e_{31}e_{123} &= e_{123}e_{31} = -e_2 \\
e_{12}e_{123} &= e_{123}e_{12} = -e_3
\end{aligned}$$

We can see that the following identification of biquaternion with $\mathrm{Cl}_{3,0}(\mathbb{R})$ elements gives exactly the same rules while obeying the biquaternion definition:

$$\begin{aligned}
i &:= -e_{23} \\
j &:= -e_{31} \\
k &:= -e_{12} \\
h &:= e_{123}
\end{aligned}$$

In particular it follows that

$$\begin{aligned}
ih &= hi = e_1 \\
jh &= hj = e_2 \\
kh &= hk = e_3
\end{aligned}$$

Thus every basis element of $\mathrm{Cl}_{3,0}(\mathbb{R})$ has a counterpart in the biquaternions.