Differential Geometry

Basics

Tangent vector: $\vec{B} = B^\mu \partial_\mu$.
Covector: $\mathbf{A} = A_\mu dx^\mu$.
Tetrad field: $\mathbf{e^a} = e^a{}_\mu dx^\mu$ ($\mathbf{e^a} = \mathbf{e^a}(x)$).
Solder form: $e = \mathbf{e^a}\gamma_a$.
$\mathbf{e^a}(\vec{B}) = e^a{}_\nu dx^\nu (B^\mu \partial_\mu) = e^a{}_\mu B^\mu = \mathbf{B^a}$.
$e(\vec{B}) = \mathbf{B^a}\gamma_a = B$.
Wedge-commutator: $[A, B]_\wedge = A \wedge B - (-1)^{gr(B)} B \wedge A$.

Forms

$\mathbf{a} \wedge \mathbf{b} = (a_\mu dx^\mu) \wedge (b_\nu dx^\nu) = \frac{1}{2}((a_\mu dx^\mu)(b_\nu dx^\nu) - (b_\nu dx^\nu)(a_\mu dx^\mu)) = a_\mu b_\nu (dx^\mu \wedge dx^\nu)$
Exterior derivative: $d = \partial_\mu dx^\mu$
Apply with wedge: $d\psi := d \wedge \psi = (\partial_\mu dx^\mu) \wedge (\psi_\nu dx^\nu) = \partial_\mu \psi_\nu (dx^\mu \wedge dx^\nu)$
$d \wedge (d \wedge \psi) = 0$.

Metric - Hodge star

$$\begin{aligned} u \cdot v &= g_{\mu\nu} u^\mu v^\nu = \eta_{ab} u^a v^b = \eta_{ab} e^a{}_\mu e^b{}_\nu u^\mu v^\nu \\ &\Rightarrow g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu = e_\mu \cdot e_\nu = g_{\nu\mu} \\ u \wedge \star v &= (u \cdot v)\ dV \end{aligned}$$

$$e^a \wedge \star e^b = \eta^{ab}\ dV \qquad e_a \wedge \star e^a = e^a \wedge \star e_a = dV$$

Connection - Derivative - Curvature

Connection: $\Omega = \frac{1}{4}\omega^{ab}\gamma_{ab} = \Omega_\circ + K$. (Levi-Civita + Contorsion)
Covariant derivative: $D = d + \Omega = D_\circ + K$
Column-action: $D\psi := D \wedge \psi = d \wedge \psi + \Omega \wedge \psi$
p-form-action: $DX := d \wedge X + \Omega \wedge X - (-1)^p X \wedge \Omega = d \wedge X + [\Omega, X]_\wedge$
Curvature:

$$\begin{aligned} D(D\psi) &= D \wedge (D \wedge \psi) \\ &= d \wedge (d \wedge \psi) + d \wedge (\Omega \wedge \psi) + \Omega \wedge (d \wedge \psi) + \Omega \wedge (\Omega \wedge \psi) \\ &= 0 + (d \wedge \Omega) \wedge \psi - \Omega \wedge (d \wedge \psi) + \Omega \wedge (d \wedge \psi) + (\Omega \wedge \Omega) \wedge \psi \\ &= (d \wedge \Omega) \wedge \psi + (\Omega \wedge \Omega) \wedge \psi \\ &= R \wedge \psi \\ R = D^2 &:= d \wedge \Omega + \Omega \wedge \Omega \end{aligned}$$

By definition: $D_\circ e := 0$
Torsion: $T := De = D_\circ e + [K, e]_\wedge = [K, e]_\wedge$
Bianchi identities:

$$\begin{aligned} DR &= d \wedge R + \Omega \wedge R - R \wedge \Omega \\ &= d \wedge (d \wedge \Omega + \Omega \wedge \Omega) + \Omega \wedge (d \wedge \Omega + \Omega \wedge \Omega) - (d \wedge \Omega + \Omega \wedge \Omega) \wedge \Omega \\ &= d \wedge (\Omega \wedge \Omega) - ((d \wedge \Omega) \wedge \Omega - \Omega \wedge (d \wedge \Omega)) \\ &= 0 \\ DT &= D(De) = D^2 e = Re := [R, e]_\wedge \end{aligned}$$

Example: Flat 2D space in polar coordinates

$x = r(\cos\theta, \sin\theta)$

$$\partial_r x = (\cos\theta, \sin\theta) \qquad \Rightarrow \quad \hat{r} = \partial_r \quad \hat{\theta} = \tfrac{1}{r}\partial_\theta \quad {e^1}_r = 1 \quad {e^2}_\theta = r$$
$$\partial_\theta x = r(-\sin\theta, \cos\theta) \qquad e = dr\ \gamma_1 + r\ d\theta\ \gamma_2$$
$$e \wedge e = 2(dr \wedge r\ d\theta)\gamma_{12}$$

Derivative:

$$d = \partial_r dr + \partial_\theta d\theta$$
$$\begin{aligned} d \wedge e &= (\partial_r 1)(dr \wedge dr)\gamma_1 + (\partial_r r)(dr \wedge d\theta)\gamma_2 + (\partial_\theta 1)(d\theta \wedge dr)\gamma_1 + (\partial_\theta r)(d\theta \wedge d\theta)\gamma_2 \\ &= (dr \wedge d\theta)\gamma_2 \end{aligned}$$

Connection:

$$\Omega = \tfrac{1}{2}\omega^{12}\gamma_{12} \qquad \omega^{12} = A\ dr + B\ d\theta$$
$$\begin{aligned} \Omega \wedge e &= \tfrac{1}{2}(\omega^{12}\gamma_{12}) \wedge (dr\ \gamma_1 + r\ d\theta\ \gamma_2) \\ &= \tfrac{1}{2}(B\gamma_2 + Ar\gamma_1)(dr \wedge d\theta) \\ &= e \wedge \Omega \end{aligned}$$

Levi-Civita:

$$\begin{aligned} De &= d \wedge e + [\Omega, e]_\wedge = d \wedge e + \Omega \wedge e + e \wedge \Omega \\ &= (\gamma_2 + B\gamma_2 + Ar\gamma_1)(dr \wedge d\theta) = 0 \end{aligned}$$
$$\Rightarrow \omega^{12} = -d\theta \qquad \Omega = -\tfrac{1}{2}d\theta\ \gamma_{12}$$

Curvature:

$$R = d \wedge \Omega + \Omega \wedge \Omega = 0$$

Example: Dirac action

$-i\mathcal{L}_D = \frac{1}{2}(\bar\psi\ e \wedge \star D\psi + \star D\bar\psi \wedge e\ \psi) + \bar\psi i m \psi\ dV$

Flat space ($D = d$):

$$
\begin{aligned}
\star D\psi &= \star d \wedge \psi = (\partial_a \psi) \star e^a \\
e \wedge \star D\psi &= (\gamma_b e^b) \wedge ((\partial_a\psi) \star e^a) \qquad\qquad (e_a \wedge \star e^a = e^a \wedge \star e_a = dV) \\
&= (\gamma^b e_b) \wedge ((\partial_a\psi) \star e^a) \\
&= (\gamma^a \partial_a \psi)\ dV \\
\star D\bar\psi \wedge e &= ((\partial_a\bar\psi) \star e^a) \wedge (\gamma^b e_b) \\
&= -(\partial_a\bar\psi)\gamma^a\ dV \\
\Rightarrow -i\mathcal{L}_D &= [\tfrac{1}{2}\bar\psi\gamma^a(\partial_a\psi) - \tfrac{1}{2}(\partial_a\bar\psi)\gamma^a\psi + \bar\psi i m\psi]\ dV \\
&\qquad\qquad \text{NB: } \partial_a(\bar\psi\gamma^a\psi) = (\partial_a\bar\psi)\gamma^a\psi + \bar\psi\gamma^a(\partial_a\psi) \\
&= [\bar\psi\gamma^a(\partial_a\psi) - \tfrac{1}{2}\partial_a(\bar\psi\gamma^a\psi) + \bar\psi i m\psi]\ dV \\
&\cong [\bar\psi\gamma^a(\partial_a\psi) + \bar\psi i m\psi]\ dV
\end{aligned}
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