e, log

Consider the derivative of a^x:

$$\begin{aligned}(a^x)' &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} \\ &= \lim_{h\to 0} a^x \frac{a^h - 1}{h} \\ &= a^x \lim_{h\to 0} \frac{a^h - 1}{h}\end{aligned}$$

Let us define $M(x) := \lim_{h\to 0} \frac{x^h-1}{h}$ and find a number e such that $M(e) = 1$. Rearranging this to find e we get

$$\begin{aligned}e &= \lim_{h\to 0}(1+h)^{\frac{1}{h}} \\ &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \\ &\approx 2.7182\end{aligned}$$

So we find $(e^x)' = e^x$ and can now consider the derivative in a different light: we know $a^x = e^{\log_e(a)x}$ so by the chain rule we now get

$$\begin{aligned}(a^x)' &= (e^{\log_e(a)x})' \\ &= \log_e(a)e^{\log_e(a)x} \\ &= \log_e(a)a^x\end{aligned}$$

Thus it follows that $\log_e(x) = M(x) = \lim_{h\to 0} \frac{x^h-1}{h}$ and we simply write $\log(x) := \log_e(x)$.

Using $\log(e^x) = x$ we can now rearrange the limit again and find an expression for e^x that only involves integer powers:

$$\begin{aligned}x &= \log(e^x) \\ &= \lim_{h\to 0} \frac{e^{xh} - 1}{h} \\ \Rightarrow & \\ e^x &= \lim_{h\to 0}(1+xh)^{\frac{1}{h}} \\ &= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\end{aligned}$$

This expression is very convenient for analyzing the behaviour of e^x when x is not simply a real number.