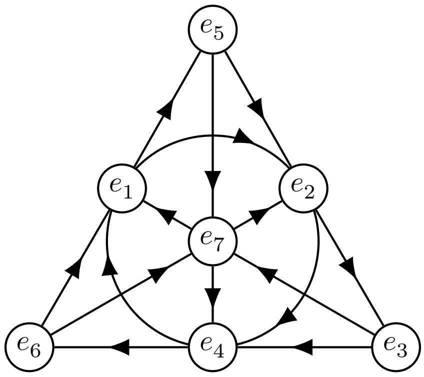


Associator:

$$
\begin{aligned}
[a, b, c] &:= (ab)c - a(bc) \\
[a + b, c, d] &= [a, c, d] + [b, c, d] \\
[a, b + c, d] &= [a, b, d] + [a, c, d] \\
[a, b, c + d] &= [a, b, c] + [a, b, d]
\end{aligned}
$$

Octonions are alternative (no proof):

$$
\begin{aligned}
[a, a, b] &= 0 \\
[a, b, b] &= 0
\end{aligned}
$$

From this follows anti-symmetry of the associator:

$$
\begin{aligned}
0 &= [a + b, a + b, c] \\
&= [a, a, c] + [a, b, c] + [b, a, c] + [b, b, c] \\
&= [a, b, c] + [b, a, c] \\
0 &= [a, b, c] + [a, c, b] \\
[a, b, c] &= -[a, c, b] \\
= -[b, a, c] &= [c, a, b] \\
= [b, c, a] &= -[c, b, a]
\end{aligned}
$$

For two anti-commuting elements x, y, such that $xy = -yx$ together with the asymmetic associator we get:

$$
\begin{aligned}
0 = [x, y, a] + [y, x, a] &= (xy)a - x(ya) + (yx)a - y(xa) \\
&= (xy)a - (xy)a - x(ya) - y(xa) \\
&= -x(ya) - y(xa) \\
x(ya) &= -y(xa) \\
0 = [a, x, y] + [a, y, x] &= (ax)y + (ay)x \\
(ay)x &= -(ax)y
\end{aligned}
$$

Putting this together, three anti-commuting and non-associative elements x, y, z anti-associate:

$$
\begin{aligned}
0 = [x, y, z] - [y, z, x] &= (xy)z - x(yz) - (yz)x + y(zx) \\
&= (xy)z - x(yz) + x(yz) + x(yz) \\
&= (xy)z + x(yz) \\
(xy)z &= -x(yz)
\end{aligned}
$$

We can label the seven imaginary units such that $e_i = e_a e_A = e_b e_B = e_c e_C$. Obviously we have:

$$[e_i, e_i] = [e_a, e_A, e_i] = [e_b, e_B, e_i] = [e_c, e_C, e_i] = 0$$

For any $e_x \neq e_i$ one of the associators vanishes, and the other two turn into the commutator with e_i due to anti-associativity and anti-commutativity. E.g. for $e_x \in \{e_b, e_B, e_c, e_C\}$:

$$\begin{aligned}[e_a, e_A, e_x] &= (e_a e_A)e_x - e_a(e_A e_x) \\ &= (e_a e_A)e_x + (e_a e_A)e_x \\ &= (e_a e_A)e_x - e_x(e_a e_A) \\ &= e_i e_x - e_x e_i \\ &= [e_i, e_x]\end{aligned}$$

Therefore the following holds for any e_x, and due to linearity of the commutator and associator for any x:

$$2[e_i, x] = [e_a, e_A, x] + [e_b, e_B, x] + [e_c, e_C, x]$$

This lets us lets us write any right-multiplication in terms of left-multiplication.

Multiplication algebras

We define left- and right-multiplication operators $L_x a = xa$ and $R_x a = ax$, which lets us rewrite the associator in different form:

$$\begin{aligned}[a, x, y] &= [R_y R_x - R_{xy}]a \\ [a, y, x] &= [R_x R_y - R_{yx}]a \\ [x, a, y] &= [R_y L_x - L_x R_y]a = [R_y, L_x]a \\ [y, a, x] &= [R_x L_y - L_y R_x]a = [R_x, L_y]a \\ [x, y, a] &= [L_{xy} - L_x L_y]a \\ [y, x, a] &= [L_{yx} - L_y L_x]a\end{aligned}$$

$$[x, [y, z]] = x[y, z] - [y, z]x = (R_{[y,z]} - L_{[y,z]})x$$

This lets us factor out a, and together with asymmetry we get:

$$\begin{aligned}R_y R_x - R_{xy} &= -R_x R_y + R_{yx} \\ &= -R_y L_x + L_x R_y \\ &= R_x L_y - L_y R_x \\ &= L_{xy} - L_x L_y \\ &= -L_{yx} + L_y L_x \\ [L_x, R_y] &= [R_x, L_y]\end{aligned}$$

Adding two of these identities on each side:

$$\begin{aligned}
R_y R_x - R_{xy} - R_x R_y + R_{yx} &= 2(R_x L_y - L_y R_x) \\
[R_y, R_x] + R_{[y,x]} &= 2[R_x, L_y] \\
[R_x, R_y] &= -R_{[x,y]} - 2[L_x, R_y]
\end{aligned}$$

$$\begin{aligned}
L_{xy} - L_x L_y - L_{yx} + L_y L_x &= 2(R_x L_y - L_y R_x) \\
[L_y, L_x] - L_{[y,x]} &= 2[R_x, L_y] \\
[L_x, L_y] &= L_{[x,y]} - 2[L_x, R_y]
\end{aligned}$$

We also get:

$$\begin{aligned}
R_x R_y + R_y R_x &= R_{xy} + R_{yx} \\
L_x L_y + L_y L_x &= L_{xy} + L_{yx}
\end{aligned}$$

TODO: show

$$\begin{aligned}
[[R_x, R_y], R_a] &= R_{[[x,y],a]-[x,y,a]+[y,x,a]-[x,a,y]+[y,a,x]-[a,x,y]+[a,y,x]} \\
&= R_{[[x,y],a]-2[x,y,a]}
\end{aligned}$$

Derivation algebra

Derivation D satisfies:

$$D(xy) = (Dx)y + x(Dy)$$

$$\begin{aligned}
D(xy) - x(Dy) &= (Dx)y \\
DL_x y - L_x Dy &= L_{Dx} y \\
[D, L_x] &= L_{Dx}
\end{aligned}$$

$$\begin{aligned}
D(xy) - (Dx)y &= x(Dy) \\
DR_y x - R_y Dx &= R_{Dy} x \\
[D, R_y] &= R_{Dy}
\end{aligned}$$

Consider

$$\begin{aligned}
D_{x,y} a :=& [[x,y],a] - 3[x,y,a] \\
=& [[x,y],a] - 3[y,a,x] \\
=& (L_{[x,y]} - R_{[x,y]} - 3[L_x, R_y])a
\end{aligned}$$

Using the identities already established we can check that it satisfies the condition above:

$$\begin{aligned}
2[D_{x,y}, R_a] &= 2[L_{[x,y]} - R_{[x,y]} - 3[L_x, R_y], R_a] \\
&= 3[-R_{[x,y]} - 2[L_x, R_y], R_a] + [R_{[x,y]} + 2L_{[x,y]}, R_a] \\
&= 3[[R_x, R_y], R_a] + [R_{[x,y]}, R_a] + 2[L_{[x,y]}, R_a] \\
&= 3[[R_x, R_y], R_a] - R_{[[x,y],a]} \\
&= 3R_{[[x,y],a]-2[x,y,a]} - R_{[[x,y],a]} \\
&= R_{2[[x,y],a]-6[x,y,a]} \\
&= 2R_{D_{x,y} a}
\end{aligned}$$

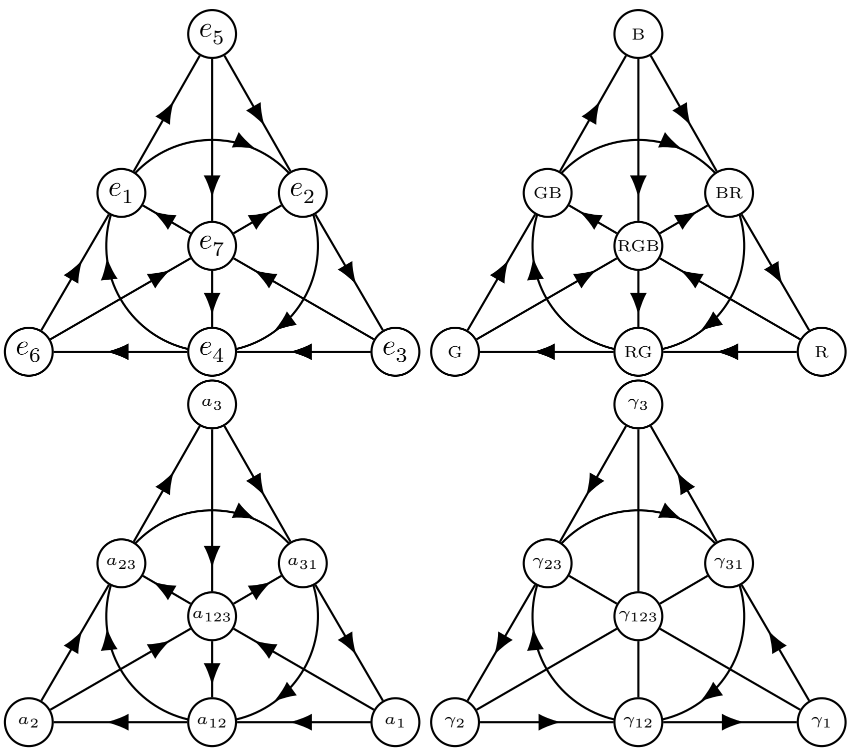
We can now express the left-right identity in terms of D-operators:

$$\begin{aligned}
2[e_i, x] &= [e_a, e_A, x] + [e_b, e_B, x] + [e_c, e_C, x] \\
6[e_i, x] &= 3[e_a, e_A, x] + 3[e_b, e_B, x] + 3[e_c, e_C, x] \\
[[e_a, e_A], x] + [[e_b, e_B], x] + [[e_c, e_C], x] &= 3[e_a, e_A, x] + 3[e_b, e_B, x] + 3[e_c, e_C, x] \\
D_{e_a,e_A}x + D_{e_b,e_B}x + D_{e_c,e_C}x = 0
\end{aligned}$$

$$\begin{array}{lll}
e_1 = e_2e_4 = e_5e_6 = e_3e_7 & \Lambda_5 = \frac{1}{6}(D_{24} - D_{56}) & g_{12} = \frac{1}{6\sqrt{3}}(D_{24} + D_{56} - 2D_{37}) \\
e_2 = e_3e_5 = e_6e_7 = e_4e_1 & \Lambda_2 = \frac{1}{6}(D_{35} - D_{41}) & g_{10} = \frac{1}{6\sqrt{3}}(D_{35} + D_{41} - 2D_{67}) \\
e_3 = e_4e_6 = e_7e_1 = e_5e_2 & \Lambda_4 = \frac{1}{6}(D_{52} - D_{46}) & g_{11} = \frac{1}{6\sqrt{3}}(D_{52} + D_{46} - 2D_{71}) \\
e_4 = e_5e_7 = e_1e_2 = e_6e_3 & \Lambda_7 = \frac{1}{6}(D_{63} - D_{12}) & g_{14} = \frac{1}{6\sqrt{3}}(D_{63} + D_{12} - 2D_{57}) \\
e_5 = e_6e_1 = e_2e_3 = e_7e_4 & \Lambda_6 = \frac{1}{6}(D_{61} - D_{23}) & g_{13} = \frac{1}{6\sqrt{3}}(D_{61} + D_{23} - 2D_{74}) \\
e_6 = e_7e_2 = e_3e_4 = e_1e_5 & \Lambda_1 = \frac{1}{6}(D_{34} - D_{15}) & g_9 = \frac{1}{6\sqrt{3}}(D_{34} + D_{15} - 2D_{72}) \\
e_7 = e_1e_3 = e_4e_5 = e_2e_6 & \Lambda_3 = \frac{1}{6}(D_{13} - D_{45}) & \Lambda_8 = -\frac{1}{6\sqrt{3}}(D_{13} + D_{45} - 2D_{26})
\end{array}$$

SU(3)

$$\begin{aligned}
e_1 &= a_3 = a_{011} = a_{GB} \\
e_2 &= -a_5 = -a_{101} = a_{BR} \\
e_3 &= a_4 = a_{100} = a_R \\
e_4 &= a_6 = a_{110} = a_{RG} \\
e_5 &= a_1 = a_{001} = a_B \\
e_6 &= a_2 = a_{010} = a_G \\
e_7 &= a_7 = a_{111} = a_{RGB}
\end{aligned}$$



$$
\begin{aligned}
\Lambda_4, a_R &: D_{B,BR} - D_{RG,G} && = -(D_{BR,B} + D_{RG,G}) \\
\Lambda_1, a_G &: D_{R,RG} - D_{GB,B} && = -(D_{RG,R} + D_{GB,B}) \\
\Lambda_6, a_B &: D_{G,GB} - D_{BR,R} && = -(D_{GB,G} + D_{BR,R}) \\
-\Lambda_5, a_{GB} &: -(D_{BR,RG} - D_{B,G}) && = -(D_{G,B} + D_{BR,RG}) \\
\Lambda_2, a_{BR} &: D_{R,B} - D_{RG,GB} && = -(D_{B,R} + D_{RG,GB}) \\
\Lambda_7, a_{RG} &: D_{G,R} - D_{GB,BR} && = -(D_{R,G} + D_{GB,BR}) \\
\Lambda_3, a_{RGB} &: D_{GB,R} - D_{RG,B} && = -(D_{R,GB} + D_{RG,B}) \\
\Lambda_8, a_{RGB} &: D_{GB,R} + D_{RG,B} - 2D_{BR,G} && = -(D_{R,GB} + D_{B,RG} + 2D_{BR,G})
\end{aligned}
$$

$$
-\Lambda_5 \cong \begin{pmatrix} 0 & & \\ & & -1 \\ & 1 & \end{pmatrix} \qquad \Lambda_2 \cong \begin{pmatrix} & & 1 \\ & 0 & \\ -1 & & \end{pmatrix} \qquad \Lambda_7 \cong \begin{pmatrix} & -1 & \\ 1 & & \\ & & 0 \end{pmatrix}
$$

$$
\Lambda_4 \cong \begin{pmatrix} 0 & & \\ & & -i \\ & -i & \end{pmatrix} \qquad \Lambda_1 \cong \begin{pmatrix} & & -i \\ & 0 & \\ -i & & \end{pmatrix} \qquad \Lambda_6 \cong \begin{pmatrix} & -i & \\ -i & & \\ & & 0 \end{pmatrix}
$$

$$
\Lambda_3 \cong \begin{pmatrix} i & & \\ & 0 & \\ & & -i \end{pmatrix} \qquad \Lambda_7 \cong \begin{pmatrix} i & & \\ & -2i & \\ & & i \end{pmatrix} \qquad v = \begin{pmatrix} R + iGB \\ G + iBR \\ B + iRG \end{pmatrix}
$$

Triality

In the following a few identities will be useful:

$$
\begin{aligned}
[x, \bar{x}, a] &= 0 & \langle abc \rangle &= \langle bca \rangle \\
a(xbx) &= [(ax)b]x & (xa)(bx) &= x(ab)x
\end{aligned}
$$

We can generate Cl(8, 0) by left-multiplication with the 8 octonionic elements:

$$
e_i = \begin{pmatrix} 0 & L_i \\ \bar{L}_i & 0 \end{pmatrix}, i \in [0, 7]
$$

Spin(8) is generated by a product of an even number of unit vectors:

$$
\begin{pmatrix} L_v L_{\bar{u}} & 0 \\ 0 & L_{\bar{v}} L_u \end{pmatrix}, u, v \in \mathbb{O}, u\bar{u} = v\bar{v} = 1
$$

Such an element acts on a pair of octonions, the two spinor representations:

$$
\begin{pmatrix} L_v L_{\bar{u}} & 0 \\ 0 & L_{\bar{v}} L_u \end{pmatrix} \begin{pmatrix} s_- \\ s_+ \end{pmatrix} = \begin{pmatrix} v(\bar{u}s_-) \\ \bar{v}(us_+) \end{pmatrix}
$$

The triality map is the real part of a triple-product $\langle \bar{s}_+ V s_- \rangle$. The vector V can be factored into a pair of spinors, so the three elements transform under Spin(8) as follows:

$$
\begin{aligned}
s_- &\to v(\bar{u}s_-) \\
\bar{s}_+ &\to (\bar{s}_+\bar{u})v \\
V = V_+\bar{V}_- &\to (\bar{v}(uV_+))((\bar{V}_-u)\bar{v}) = \bar{v}(uVu)\bar{v}
\end{aligned}
$$

The triality product is indeed symmetric under the given Spin(8) action:

$$\begin{aligned}
\langle \bar{s}_+ V s_- \rangle &\to \langle [(\bar{s}_+ \bar{u})v]\,[\bar{v}(uVu)\bar{v}]\,[v(\bar{u}s_-)] \rangle \\
&= \langle [([(\bar{s}_+ \bar{u})v]\,\bar{v})\,(uVu)]\,\bar{v}\,[v(\bar{u}s_-)] \rangle \\
&= \langle (\bar{s}_+ \bar{u})(uVu)(\bar{u}s_-) \rangle \\
&= \langle ([(\bar{s}_+ \bar{u})u]\,V)\,u(\bar{u}s_-) \rangle \\
&= \langle \bar{s}_+ V s_- \rangle
\end{aligned}$$

Decomposing $V = V_+ \bar{V}_-$ leads to an interesting observation:

$$\begin{aligned}
\langle \bar{s}_+ V s_- \rangle &= \langle \bar{s}_+ (V_+ \bar{V}_-) s_- \rangle \\
&= \langle s_- \bar{s}_+ (V_+ \bar{V}_-) \rangle \\
&= \langle (s_- \bar{s}_+) V_+ \bar{V}_- \rangle \\
&= \langle \bar{V}_- (s_- \bar{s}_+) V_+ \rangle
\end{aligned}$$

Cayley-Dickson product

Canonical, nice form of $\bar{a}b$:

$$\begin{aligned}
(a,b)(c,d) &:= (ac - \bar{d}b, da + b\bar{c}) \\
\overline{(a,b)}(c,d) &:= (\bar{a}c + \bar{d}b, d\bar{a} - b\bar{c})
\end{aligned}$$

Alternative, nice form of $a\bar{b}$:

$$\begin{aligned}
(a,b)(c,d) &:= (ac - d\bar{b}, \bar{a}d + cb) \\
(a,b)\overline{(c,d)} &:= (a\bar{c} + d\bar{b}, -\bar{a}d + \bar{c}b)
\end{aligned}$$

Misc

TODO: moufang identities, alternativity, flexibility (general), power associativity (general)

TODO: $x(x^* a) = (xx^*)a$.

TODO: Spin(8) triality symmetry