Quaternions

Quaternions are the 3D equivalent to complex numbers. They are made up of four numbers (whence the name) but do *not* describe 4D space.

We introduce three elements i, j, k that obey the following rules:

$$\begin{aligned} i^2 = j^2 = k^2 &= -1 \\ jk = -kj &= i \\ ij = -ij &= k \\ ki = -ik &= j \end{aligned}$$

A general quaternion will be denoted as $q = q_0 + q_1 i + q_2 j + q_3 k = q_0 + \mathbf{q_v}$, where $\mathbf{q_v}$ is a quaternionic 3D vector I will call a qvector. The product of two quaternions q and p looks as follows:

$$\begin{aligned} qp &= (q_0 + \mathbf{q_v})(p_0 + \mathbf{p_v}) \\ &= q_0 p_0 + q_0 \mathbf{p_v} + p_0 \mathbf{q_v} + \mathbf{q_v}\mathbf{p_v} \end{aligned}$$

where

$$\begin{aligned} \mathbf{q_v}\mathbf{p_v} &= -q_1 p_1 - q_2 p_2 - q_3 p_3 + i(q_2 p_3 - q_3 p_2) + j(q_3 p_1 - q_1 p_3) + k(q_1 p_2 - q_2 p_1) \\ &= -\mathbf{q_v} \cdot \mathbf{p_v} + \mathbf{q_v} \times \mathbf{p_v} \end{aligned}$$

with

$$\begin{aligned} \mathbf{q_v} \cdot \mathbf{p_v} &= q_1 p_1 + q_2 p_2 + q_3 p_3 \\ \mathbf{q_v} \times \mathbf{p_v} &= i(q_2 p_3 - q_3 p_2) \\ &\quad + j(q_3 p_1 - q_1 p_3) \\ &\quad + k(q_1 p_2 - q_2 p_1) \end{aligned}$$

which we call the dot-product and cross-product respectively. Note that

$$\begin{aligned} \mathbf{q_v} \cdot \mathbf{p_v} &= \mathbf{p_v} \cdot \mathbf{q_v} \\ \mathbf{q_v} \times \mathbf{p_v} &= -\mathbf{p_v} \times \mathbf{q_v} \\ \mathbf{q_v} \times \mathbf{q_v} &= 0 \end{aligned}$$

which allows us to express both as the symmetric and anti-symmetric part of the general product:

$$\begin{aligned} \mathbf{q_v} \cdot \mathbf{p_v} &= -\frac{1}{2}(\mathbf{q_v}\mathbf{p_v} + \mathbf{p_v}\mathbf{q_v}) \\ \mathbf{q_v} \times \mathbf{p_v} &= \frac{1}{2}(\mathbf{q_v}\mathbf{p_v} - \mathbf{p_v}\mathbf{q_v}) \end{aligned}$$

The conjugate we will denote with $q^* := q_0 - \mathbf{q_v}$ and the squared norm with $|q|^2 := qq^* = q_0^2 + q_1^2 + q_2^2 + q_3^2$. For the conjugate of a product we find:

$$\begin{aligned} (qp)^* &= ((q_0 + \mathbf{q_v})(p_0 + \mathbf{p_v}))^* \\ &= (q_0 p_0 + q_0 \mathbf{p_v} + p_0 \mathbf{q_v} - \mathbf{q_v} \cdot \mathbf{p_v} + \mathbf{q_v} \times \mathbf{p_v})^* \\ &= q_0 p_0 - q_0 \mathbf{p_v} - p_0 \mathbf{q_v} - \mathbf{q_v} \cdot \mathbf{p_v} - \mathbf{q_v} \times \mathbf{p_v} \\ &= p_0 q_0 - p_0 \mathbf{q_v} - q_0 \mathbf{p_v} - \mathbf{p_v} \cdot \mathbf{q_v} + \mathbf{p_v} \times \mathbf{q_v} \\ &= (p_0 - \mathbf{p_v})(q_0 - \mathbf{q_v}) \\ &= p^* q^* \end{aligned}$$

With this we get the important property that norms multiply:

$$\begin{aligned}|qp|^2 &= (qp)(qp)^* \\ &= (qp)(p^*q^*) = q(pp^*)q^* \\ &= q|p|^2q^* = qq^*|p|^2 \\ &= |q|^2|p|^2\end{aligned}$$

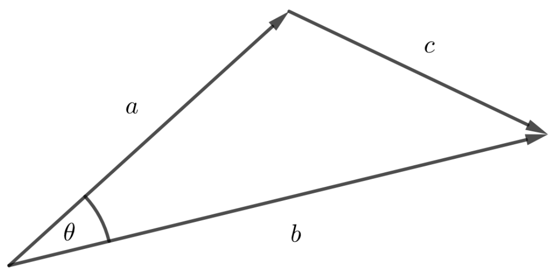
With $q = |q|\hat{q}$ where $\hat{q}$ is a unit-quaternion this property gives us

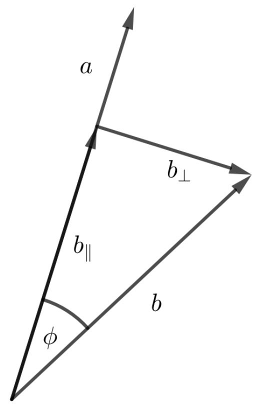
$$\begin{aligned}(|\mathbf{a}|\hat{\mathbf{a}}) \cdot (|\mathbf{b}|\hat{\mathbf{b}}) &= -\frac{1}{2}(|\mathbf{a}||\mathbf{b}|\hat{\mathbf{a}}\hat{\mathbf{b}} + |\mathbf{a}||\mathbf{b}|\hat{\mathbf{b}}\hat{\mathbf{a}}) \\ &= |\mathbf{a}||\mathbf{b}|(-\frac{1}{2}(\hat{\mathbf{a}}\hat{\mathbf{b}} + \hat{\mathbf{b}}\hat{\mathbf{a}})) \\ &= |\mathbf{a}||\mathbf{b}|(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}})\end{aligned}$$

and similarly

$$(|\mathbf{a}|\hat{\mathbf{a}}) \times (|\mathbf{b}|\hat{\mathbf{b}}) = |\mathbf{a}||\mathbf{b}|(\hat{\mathbf{a}} \times \hat{\mathbf{b}})$$

Let us now consider the geometric interpretation of these products.





Let $\mathbf{a}$ and $\mathbf{b}$ be any qvectors and $\mathbf{c} = \mathbf{b} - \mathbf{a}$. If we square both sides we get:

$$\begin{aligned}\mathbf{c}^2 &= (\mathbf{b} - \mathbf{a})^2 \\ &= \mathbf{b}^2 + \mathbf{a}^2 - \mathbf{ab} - \mathbf{ba} \\ -|\mathbf{c}|^2 &= -|\mathbf{b}|^2 - |\mathbf{a}|^2 + 2\mathbf{a} \cdot \mathbf{b} \\ |\mathbf{c}|^2 &= |\mathbf{b}|^2 + |\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b}\end{aligned}$$

and the Pythagorean theorem implies

$$\mathbf{a} \perp \mathbf{b} \iff \mathbf{a} \cdot \mathbf{b} = 0$$

Now consider the product $\mathbf{a} \cdot \mathbf{b}$ in general. We can decompose $\mathbf{b}$ into components parallel and perpendicular to $\mathbf{a}$ and we find the following:

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= \mathbf{a} \cdot (\mathbf{b}_{\parallel} + \mathbf{b}_{\perp}) = \mathbf{a} \cdot \mathbf{b}_{\parallel} + \mathbf{a} \cdot \mathbf{b}_{\perp} \\ &= \mathbf{a} \cdot \mathbf{b}_{\parallel} + 0 \\ &= \mathbf{a} \cdot \frac{\mathbf{a}}{|\mathbf{a}|}|\mathbf{b}_{\parallel}| \\ &= \frac{\mathbf{a}^2}{|\mathbf{a}|}|\mathbf{b}|\cos(\phi) = \frac{|\mathbf{a}|^2}{|\mathbf{a}|}|\mathbf{b}|\cos(\phi) = |\mathbf{a}||\mathbf{b}|\cos(\phi)\end{aligned}$$

For the cross product we have

$$\mathbf{a} \parallel \mathbf{b} \Longleftrightarrow \mathbf{a} \times \mathbf{b} = 0$$

and we can again decompose $\mathbf{b}$:

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \mathbf{a} \times (\mathbf{b}_{\parallel} + \mathbf{b}_{\perp}) = \mathbf{a} \times \mathbf{b}_{\parallel} + \mathbf{a} \times \mathbf{b}_{\perp} \\
&= 0 + \mathbf{a} \times \mathbf{b}_{\perp} \\
&= \frac{\mathbf{a}}{|\mathbf{a}|} \times \frac{\mathbf{b}_{\perp}}{|\mathbf{b}_{\perp}|} |\mathbf{a}||\mathbf{b}_{\perp}| \\
&= \frac{\mathbf{a} \times \mathbf{b}_{\perp}}{|\mathbf{a}\mathbf{b}_{\perp}|} |\mathbf{a}||\mathbf{b}| \sin(\phi) \\
&= \frac{\mathbf{a}\mathbf{b}_{\perp}}{|\mathbf{a}\mathbf{b}_{\perp}|} |\mathbf{a}||\mathbf{b}| \sin(\phi)
\end{aligned}$$

The first part of this is a unit qvector, but what about its direction? We can easily show that it is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$:

$$\begin{aligned}
(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} &= \frac{1}{2}(\mathbf{ab} - \mathbf{ba}) \cdot \mathbf{a} \\
&= \frac{1}{4}(\mathbf{aab} - \mathbf{aba} + \mathbf{aba} - \mathbf{baa}) \\
&= \frac{1}{4}(\mathbf{a}^2\mathbf{b} - \mathbf{ba}^2) \\
&= \frac{1}{4}|\mathbf{a}|^2(\mathbf{b} - \mathbf{b}) = 0 \\
(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} &= ... \\
&= 0
\end{aligned}$$