U(1)

$$\mathcal{L}_{Dirac} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

Global U(1) symmetry:

$$\begin{aligned}
\psi \to \psi' &= e^{iq\theta}\psi \\
\mathcal{L}'_{Dirac} &= \bar{\psi}e^{-iq\theta}(i\gamma^\mu \partial_\mu - m)(e^{iq\theta}\psi) \\
&= \bar{\psi}e^{-iq\theta}e^{iq\theta}(i\gamma^\mu \partial_\mu - m)\psi \\
&= \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi
\end{aligned}$$

Want local U(1) symmetry: $\theta \to \theta(x)$. $\partial_\mu \theta \neq 0$.

$$\begin{aligned}
\psi \to \psi' &= e^{iq\theta}\psi \\
\partial_\mu(e^{iq\theta}\psi) &= \partial_\mu(e^{iq\theta})\psi + e^{iq\theta}\partial_\mu\psi \\
&= iq(\partial_\mu\theta)e^{iq\theta}\psi + e^{iq\theta}\partial_\mu\psi \\
&= e^{iq\theta}(\partial_\mu + iq\partial_\mu\theta)\psi
\end{aligned}$$

$$\begin{aligned}
D_\mu &:= \partial_\mu + iqA_\mu \\
A_\mu \to A'_\mu &= A_\mu - \partial_\mu\theta \\
D'_\mu(e^{iq\theta}\psi) &= e^{iq\theta}(\partial_\mu + iq\partial_\mu\theta + iqA'_\mu)\psi \\
&= e^{iq\theta}(\partial_\mu + iq\partial_\mu\theta + iqA_\mu - iq\partial_\mu\theta)\psi \\
&= e^{iq\theta}(\partial_\mu + iqA_\mu)\psi \\
&= e^{iq\theta}D_\mu\psi
\end{aligned}$$

$$\begin{aligned}
\mathcal{L} &= \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \\
\mathcal{L}' &= \bar{\psi}e^{-iq\theta}(i\gamma^\mu D'_\mu - m)(e^{iq\theta}\psi) \\
&= \bar{\psi}e^{-iq\theta}e^{iq\theta}(i\gamma^\mu D_\mu - m)\psi \\
&= \bar{\psi}(i\gamma^\mu D_\mu - m)\psi = \mathcal{L}
\end{aligned}$$

Add kinetic term for A by hand:

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\
F'_{\mu\nu} &= \partial_\mu(A_\nu - \partial_\nu\theta) - \partial_\nu(A_\mu - \partial_\mu\theta) \\
&= \partial_\mu A_\nu - \partial_\mu\partial_\nu\theta - \partial_\nu A_\mu + \partial_\nu\partial_\mu\theta \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} \\
\mathcal{L}_{QED} &= \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
&= \bar{\psi}(i\gamma^\mu \partial_\mu - q\gamma^\mu A_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}
\end{aligned}$$

General Covariant Derivative

Assume transformation $\psi \to \psi' = U\psi$. Want $D'_\mu(\psi') = D'_\mu(U\psi) = UD_\mu(\psi)$ with $D_\mu = \partial_\mu + X_\mu$. Goal: find X'_μ.

$$\begin{aligned}
\partial_\mu(U\psi) &= (\partial_\mu U)\psi + U(\partial_\mu \psi) \\
D'_\mu(U\psi) &= \partial_\mu(U\psi) + X'_\mu U\psi \\
&= (\partial_\mu U)\psi + U(\partial_\mu \psi) + X'_\mu U\psi \\
&= UD_\mu(\psi) = U(\partial_\mu \psi) + UX_\mu\psi
\end{aligned}$$

$$\begin{aligned}
UX_\mu &= (\partial_\mu U) + X'_\mu U \\
X'_\mu &= UX_\mu U^{-1} - (\partial_\mu U)U^{-1}
\end{aligned}$$

$\mathbf{SU(2)_W}$ and $\mathbf{U(1)_Y}$

$\mathrm{SU}(2)_W$:

$$\begin{aligned}
U &= e^{ig_w \tau^a \theta^a(x)} \\
X &= -ig_w \tau^a W^a_\mu \\
D_\mu &= \partial_\mu - ig_w \tau^a W^a_\mu
\end{aligned}$$

$\mathrm{U}(1)_Y$:

$$\begin{aligned}
U &= e^{ig_y Y\theta(x)} \\
X &= -ig_y Y B_\mu \\
D_\mu &= \partial_\mu - ig_y Y B_\mu
\end{aligned}$$

Higgs

$D_\mu = \partial_\mu + ig_w \tau^a W^a_\mu + ig_y Y B_\mu$, where the τ are the σ-matrices but in weak flavor-space, and Y = 1 for the Higgs-field. We investigate the case of a non-zero vacuum expectation value, which we can arbitraryly choose to be a scalar isospin-down state: a global phase cancels out and the choice of isospin-down simply picks out a specific but arbitrary direction, which we will consider to be aligned with W^3. We will use $\varphi_0 = \begin{pmatrix} 0 \\ v \end{pmatrix}$. Note that $\tau_1\varphi_0 = i\tau_2\varphi_0$, $\tau_3\varphi_0 = -\varphi_0$, $\tau_1^\dagger = \tau_1$, $\varphi_0^\dagger \tau_1 \varphi_0 = 0$, $\varphi^\dagger\varphi = |\varphi|^2$.

$$\begin{aligned}
D_\mu\varphi_0 &= \partial_\mu\varphi_0 + ig_w\tau^a W^a_\mu \varphi_0 + ig_y B_\mu \varphi_0 \\
&= \partial_\mu\varphi_0 + g_w(i\tau_1 W^1_\mu\varphi_0 + i\tau_2 W^2_\mu\varphi_0 + i\tau_3 W^3_\mu\varphi_0) + g_y B_\mu(i\varphi_0) \\
&= \partial_\mu\varphi_0 + g_w(W^1_\mu(i\tau_1\varphi_0) + W^2_\mu(\tau_1\varphi_0) + W^3_\mu(-i\varphi_0)) + g_y B_\mu(i\varphi_0) \\
&= \partial_\mu\varphi_0 + g_w(iW^1_\mu + W^2_\mu)\tau_1\varphi_0 + i(g_y B_\mu - g_w W^3_\mu)\varphi_0 \\
(D_\mu\varphi_0)^\dagger &= \partial_\mu\varphi_0^\dagger + g_w(W^1_\mu(-i\varphi_0^\dagger i\tau_1) + W^2_\mu(\varphi_0^\dagger\tau_1) + W^3_\mu(i\varphi_0^\dagger)) + g_y B_\mu(-i\varphi_0) \\
&= \partial_\mu\varphi_0^\dagger + g_w(-iW^1_\mu + W^2_\mu)\varphi_0^\dagger\tau_1 - i(g_y B_\mu - g_w W^3_\mu)\varphi_0^\dagger
\end{aligned}$$

Because the W^1 and W^2 terms are orthogonal to the ∂_μ, W^3 and B terms, a lot of the products become zero, and some cancel out.

$$\begin{aligned}
(D_\mu\varphi_0)^\dagger(D^\mu\varphi_0) =&(\partial_\mu\varphi_0^\dagger)(\partial^\mu\varphi_0)\\
&+g_w^2|\varphi_0|^2(W_\mu^1+iW_\mu^2)(W^{1\mu}-iW^{2\mu})\\
&+|\varphi_0|^2(g_wW_\mu^3-g_yB_\mu)(g_wW^{3\mu}-g_yB^\mu)\\
=&(\partial_\mu\varphi_0^\dagger)(\partial^\mu\varphi_0)\\
&+g_w^2|\varphi_0|^2W_\mu^1W^{1\mu}+g_w^2|\varphi_0|^2W_\mu^2W^{2\mu}\\
&+g_w^2|\varphi_0|^2W_\mu^3W^{3\mu}+g_y^2|\varphi_0|^2B_\mu B^\mu-2g_wg_y|\varphi_0|^2W_\mu^3B^\mu
\end{aligned}$$

Where the $W^{1/2}$ terms have been multiplied by $(-i)(i)$ to give the W^+ and W^- bosons. For the W^3 and B terms we first rotate them into a superposition by angle θ_w:

$$\begin{aligned}
\begin{pmatrix}W^3\\B\end{pmatrix} &= \begin{pmatrix}\cos\theta_w & \sin\theta_w\\ -\sin\theta_w & \cos\theta_w\end{pmatrix}\begin{pmatrix}Z\\A\end{pmatrix}\\
g_wW_\mu^3-g_yB_\mu &= g_w\cos\theta_wZ_\mu+g_w\sin\theta_wA_\mu+g_y\sin\theta_wZ_\mu-g_y\cos\theta_wA_\mu\\
&=(g_w\sin\theta_w-g_y\cos\theta_w)A_\mu+(g_w\cos\theta_w+g_y\sin\theta_w)Z_\mu
\end{aligned}$$

For $\tan\theta_w=\frac{g_y}{g_w}$ the A term vanishes and we can rewrite $(g_w\cos\theta_w+g_y\sin\theta_w)^2=g_w^2+g_y^2$.

$$\begin{aligned}
W^+ &:= \frac{1}{\sqrt{2}}(W^1+iW^2)\\
W^- &:= \frac{1}{\sqrt{2}}(W^1-iW^2)\\
(D_\mu\varphi_0)^\dagger(D^\mu\varphi_0) =&(\partial_\mu\varphi_0^\dagger)(\partial^\mu\varphi_0)\\
&+2|\varphi_0|^2g_w^2W_\mu^+W^{-\mu}\\
&+|\varphi_0|^2(g_w^2+g_y^2)Z_\mu Z^\mu
\end{aligned}$$