Pauli Spinor

In Cl(3,0), define

$$
\begin{aligned}
|\uparrow\rangle &:= \frac{1}{2}(1+\sigma_z) \\
|\downarrow\rangle &:= \sigma_x |\uparrow\rangle
\end{aligned}
$$

NB:

$$
\begin{aligned}
&|\uparrow\rangle |\uparrow\rangle = |\uparrow\rangle \\
&|\downarrow\rangle |\downarrow\rangle = |\uparrow\rangle |\downarrow\rangle = |\downarrow\rangle |\uparrow\rangle = 0 \\
&\sigma_z |\uparrow\rangle = + |\uparrow\rangle \\
&\sigma_z |\downarrow\rangle = - |\downarrow\rangle
\end{aligned}
$$

Let $q \in \mathbb{H}, q = a_0 + a_x\sigma_{yz} + a_y\sigma_{zx} + a_z\sigma_{xy}, |q| = 1$:

$$
\begin{aligned}
q |\uparrow\rangle =& (a_0 + a_z\sigma_{xy}) |\uparrow\rangle + (a_x\sigma_y - a_y\sigma_x)\sigma_z |\uparrow\rangle \\
=& (a_0 + a_z\sigma_{xy}) |\uparrow\rangle - (a_y + a_x\sigma_{xy})\sigma_x |\uparrow\rangle \\
=& (a_0 + a_z\sigma_{xy}) |\uparrow\rangle - (a_y + a_x\sigma_{xy}) |\downarrow\rangle \\
=& \cos\frac{\phi}{2} e^{\frac{1}{2}\theta_\uparrow\sigma_{xy}} |\uparrow\rangle - \sin\frac{\phi}{2} e^{\frac{1}{2}\theta_\downarrow\sigma_{xy}} |\downarrow\rangle
\end{aligned}
$$

Thus

$$
\begin{aligned}
a_0 &= \cos\frac{\phi}{2}\cos\frac{\theta_\uparrow}{2} \\
a_x &= \sin\frac{\phi}{2}\sin\frac{\theta_\downarrow}{2} \\
a_y &= \sin\frac{\phi}{2}\cos\frac{\theta_\downarrow}{2} \\
a_z &= \cos\frac{\phi}{2}\sin\frac{\theta_\uparrow}{2}
\end{aligned}
$$

If we identify the complex i with σ_{xy}, then $e^{\theta\sigma_{xy}}$ is simply a phase factor. In particular, let $\theta_\uparrow = 0$, then ϕ is a polar and $\theta_\downarrow$ an azimuthal angle:

$$
\begin{aligned}
a_0 &= \cos\frac{\phi}{2} \\
a_x &= \sin\frac{\phi}{2}\sin\frac{\theta_\downarrow}{2} \\
a_y &= \sin\frac{\phi}{2}\cos\frac{\theta_\downarrow}{2} \\
a_z &= 0
\end{aligned}
$$

Dirac Spinor

Full STA (= Cl(1,3)) using Dirac basis:

$$1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \gamma_{0123} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma_{123} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_{230} = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_{310} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \gamma_{120} = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

$$\gamma_{23} = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix} \qquad \gamma_{10} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_{31} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \gamma_{20} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_{12} = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \qquad \gamma_{30} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4}(1 + i\gamma_{12})(1 + \gamma_0) = \frac{1}{4}(1 + i\gamma_{120})(1 + \gamma_0)$$

Dirac left column basis:

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}:1, \gamma_0 \qquad \begin{pmatrix}i\\0\\0\\0\end{pmatrix}: -\gamma_{12}, -\gamma_{120}$$

$$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}:\gamma_{31}, \gamma_{310} \qquad \begin{pmatrix}0\\i\\0\\0\end{pmatrix}: -\gamma_{23}, -\gamma_{230}$$

$$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}: -\gamma_{30}, -\gamma_3 \qquad \begin{pmatrix}0\\0\\i\\0\end{pmatrix}: -\gamma_{0123}, \gamma_{123}$$

$$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}: -\gamma_{10}, -\gamma_1 \qquad \begin{pmatrix}0\\0\\0\\i\end{pmatrix}: -\gamma_{20}, -\gamma_2$$

Note that such a column vector can always be interpreted as the projection of an even STA multivector (= Lorentz transformation). In particular, the upper two entries are a quaternion, and the lower two are a quaternion multiplied by $-\gamma_{30}/-\gamma_3$. The right side (imaginary) is the left side (real) multiplied by $-\gamma_{12}$.

Chiral basis:

$$1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \gamma_{0123} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

$$\gamma_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \gamma_{123} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_{230} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_{310} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \gamma_{120} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\gamma_{23} = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix} \qquad \gamma_{10} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\gamma_{31} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \gamma_{20} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}$$

$$\gamma_{12} = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \qquad \gamma_{30} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4}(1 + i\gamma_{12})(1 + \gamma_{30}) = \frac{1}{4}(1 - i\gamma_{0123})(1 + \gamma_{30})$$

$$\sigma_1 = i\rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = i\rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = i\rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \gamma_{0123} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_{123} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\gamma_1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \qquad \gamma_{230} = \begin{pmatrix} 0 & -i\sigma_1 \\ -i\sigma_1 & 0 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \qquad \gamma_{310} = \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} \qquad \gamma_{120} = \begin{pmatrix} 0 & -i\sigma_3 \\ -i\sigma_3 & 0 \end{pmatrix}$$

$$\gamma_{23} = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix} \qquad \gamma_{10} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}$$

$$\gamma_{31} = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} \qquad \gamma_{20} = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}$$

$$\gamma_{12} = \begin{pmatrix} -i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{pmatrix} \qquad \gamma_{30} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}$$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \gamma_{0123} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_{123} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\gamma_1 = \begin{pmatrix} 0 & i\rho_1 \\ -i\rho_1 & 0 \end{pmatrix} \qquad \gamma_{230} = \begin{pmatrix} 0 & \rho_1 \\ \rho_1 & 0 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} 0 & i\rho_2 \\ -i\rho_2 & 0 \end{pmatrix} \qquad \gamma_{310} = \begin{pmatrix} 0 & \rho_2 \\ \rho_2 & 0 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 0 & i\rho_3 \\ -i\rho_3 & 0 \end{pmatrix} \qquad \gamma_{120} = \begin{pmatrix} 0 & \rho_3 \\ \rho_3 & 0 \end{pmatrix}$$

$$\gamma_{23} = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_1 \end{pmatrix} \qquad \gamma_{10} = \begin{pmatrix} i\rho_1 & 0 \\ 0 & -i\rho_1 \end{pmatrix}$$

$$\gamma_{31} = \begin{pmatrix} \rho_2 & 0 \\ 0 & \rho_2 \end{pmatrix} \qquad \gamma_{20} = \begin{pmatrix} i\rho_2 & 0 \\ 0 & -i\rho_2 \end{pmatrix}$$

$$\gamma_{12} = \begin{pmatrix} \rho_3 & 0 \\ 0 & \rho_3 \end{pmatrix} \qquad \gamma_{30} = \begin{pmatrix} i\rho_3 & 0 \\ 0 & -i\rho_3 \end{pmatrix}$$

Chiral left column basis:

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} : 1, \gamma_{30} \qquad \begin{pmatrix}i\\0\\0\\0\end{pmatrix} : -\gamma_{12}, \gamma_{0123}$$

$$\begin{pmatrix}0\\1\\0\\0\end{pmatrix} : \gamma_{31}, \gamma_{10} \qquad \begin{pmatrix}0\\i\\0\\0\end{pmatrix} : -\gamma_{23}, \gamma_{20}$$

$$\begin{pmatrix}0\\0\\1\\0\end{pmatrix} : -\gamma_{3}, \gamma_{0} \qquad \begin{pmatrix}0\\0\\i\\0\end{pmatrix} : \gamma_{123}, -\gamma_{120}$$

$$\begin{pmatrix}0\\0\\0\\1\end{pmatrix} : -\gamma_{1}, \gamma_{310} \qquad \begin{pmatrix}0\\0\\0\\i\end{pmatrix} : -\gamma_{2}, -\gamma_{230}$$

Note that such a column vector can always be interpreted as the projection of a multivector of the Cl(0,3) subalgebra of STA. In particular, the upper two entries (the left chiral spinor) can be interpreted as the projection of an even element of STA and are represented by a quaternion. The lower two entries (the right chiral spinor) are the same but multiplied by $\gamma_0 / -\gamma_3$. The right side (imaginary) is the left side (real) multiplied by $-\gamma_{12}$.

We define the following:

$$\begin{aligned} P_1^{\pm} &:= \frac{1}{2}(1 \mp i\gamma_{0123}) \\ P_2^{\pm} &:= \frac{1}{2}(1 \pm \gamma_{30}) \\ P &:= P_1^{+} P_2^{+} \end{aligned}$$

Then a spinor can be thought of as the sum of a left and right chiral spinor, $\psi_{1,2} \in \mathbb{H}$ ($=$ span($\{1, \gamma_{23}, \gamma_{31}, \gamma_{12}\}$)):

$$\psi = (\psi_1 + \gamma_0 \psi_2) P$$

Note that due to P each quaternion stays a quaternion under multiplication by an even element L of STA (Lorentz transformations):

$$L(\psi_i P) = (L\psi_i) P = \psi_i' P$$

For this reason no Lorentz-transformation can mix ψ_1 and ψ_2. Multiplication by various γ_μ yields:

$$\begin{aligned} \gamma_0 \psi &= (\gamma_0 \psi_1 + \gamma_0 \gamma_0 \psi_2) P = (\psi_2 + \gamma_0 \psi_1) P \\ \gamma_i \psi &= (\gamma_i \psi_1 + \gamma_i \gamma_0 \psi_2) P = (\gamma_{i0} \psi_2 - \gamma_0 \gamma_{i0} \psi_1) P \\ \gamma_{ij} \psi &= (\gamma_{ij} \psi_1 + \gamma_{ij} \gamma_0 \psi_2) P = (\gamma_{ij} \psi_1 + \gamma_0 \gamma_{ij} \psi_2) P \\ \gamma_{i0} \psi &= (\gamma_{i0} \psi_1 + \gamma_{i0} \gamma_0 \psi_2) P = (\gamma_{i0} \psi_1 - \gamma_0 \gamma_{i0} \psi_2) P \end{aligned}$$

Note that both spinors behave the same under multiplication by a Pauli-even element (rotations) but with opposite sign under a Pauli-odd element (boosts). γ_0 flips left and right spinors.

To define a Lorentz-invariant inner product, Hermitian conjugation can be expressed as

$$
\begin{aligned}
a^\dagger &= \gamma_0 \tilde{a} \gamma_0 \\
\tilde{i} &= -i \qquad\qquad \text{(for } P_1\text{)}
\end{aligned}
$$

and thus:

$$
\begin{aligned}
\gamma_0 \tilde{P} \gamma_0 &= P (\neq \tilde{P}!) \\
\gamma_0 \tilde{\psi} \gamma_0 &= P \gamma_0 (\tilde{\psi_1} + \tilde{\psi_2} \gamma_0) \gamma_0 = P(\tilde{\psi_1} + \gamma_0 \tilde{\psi_2})
\end{aligned}
$$

Some properties of the projectors are:

$$
\begin{aligned}
P_i^2 &= P_i \\
P_i^+ P_i^- &= P_i^- P_i^+ = 0 \\
P_1^\pm \gamma_\mu &= \gamma_\mu P_1^\mp \\
P \gamma_\mu P &= 0
\end{aligned}
$$

The product of a spinor and its conjugate:

$$
\begin{aligned}
\psi^\dagger \psi &= P(\tilde{\psi_1} + \gamma_0 \tilde{\psi_2})(\psi_1 + \gamma_0 \psi_2) P \\
&= P[(\tilde{\psi_1} \psi_1 + \tilde{\psi_2} \psi_2) + \gamma_0 (\tilde{\psi_1} \psi_2 + \tilde{\psi_2} \psi_1)] P \\
&= P[(|\psi_1|^2 + |\psi_2|^2) + \gamma_0 (\tilde{\psi_1} \psi_2 + \widetilde{\tilde{\psi_1} \psi_2})] P \\
&= P(a + \gamma_0 b) P = P a P = a P
\end{aligned}
$$

with $a, b \in \mathbb{R}$ because the sum of a quaternion and its conjugate/reverse is real. This product is however not yet Lorentz-invariant. Any Lorentz-transformation can be expressed as the product of a commuting rotation and boost:

$$
\psi \to \psi' = e^{aA} e^{bB} \psi
$$

where A and B are bivectors generating a rotation and boost respectively with these properties:

$$
\begin{aligned}
&AB = BA \\
&A^2 = -1 \qquad\qquad \gamma_0 A = A \gamma_0 \qquad\qquad A^\dagger = -A \\
&B^2 = 1 \qquad\qquad \gamma_0 B = -B \gamma_0 \qquad\qquad B^\dagger = B
\end{aligned}
$$

The conjugate spinor transforms as follows:

$$
\begin{aligned}
\psi^\dagger \to \psi'^\dagger &= \psi^\dagger (e^{bB})^\dagger (e^{aA})^\dagger \\
&= \psi^\dagger e^{bB} e^{-aA}
\end{aligned}
$$

Clearly this is missing a negation of B to cancel out the transformation on ψ. So we define the Dirac adjoint:

$$
\bar{\psi} = \psi^\dagger \gamma_0
$$

For the inner product we now get:

$$
\bar{\psi} \psi = ... = P \gamma_0 (a + \gamma_0 b) P = P(b + \gamma_0 a) P = P b P = b P
$$

with a, b from above. And:

$$\begin{aligned}\bar{\psi} \to \bar{\psi}' = \psi'^{\dagger}\gamma_0 &= \psi^{\dagger}(e^{bB})^{\dagger}(e^{aA})^{\dagger}\gamma_0 \\ &= \psi^{\dagger}e^{bB}e^{-aA}\gamma_0 \\ &= \psi^{\dagger}e^{bB}\gamma_0 e^{-aA} \\ &= \psi^{\dagger}\gamma_0 e^{-bB}e^{-aA} \\ &= \bar{\psi}e^{-bB}e^{-aA}\end{aligned}$$

thus the transformations cancel out to give:

$$\bar{\psi}'\psi' = \bar{\psi}\psi$$

Majorana basis:

$$1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \gamma_{0123} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\gamma_0 = i \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_{123} = i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\gamma_1 = i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma_{230} = i \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\gamma_2 = i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_{310} = i \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\gamma_3 = i \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad \gamma_{120} = i \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma_{23} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \gamma_{10} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_{31} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \gamma_{20} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\gamma_{12} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_{30} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4}(1 - i\gamma_1)(1 - \gamma_{20}) = \frac{1}{4}(1 + i\gamma_{120})(1 - \gamma_{20})$$

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}:1,-\gamma_{20} \qquad \begin{pmatrix}i\\0\\0\\0\end{pmatrix}:-\gamma_{120},\gamma_{1}$$

$$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}:\gamma_{31},\gamma_{0123} \qquad \begin{pmatrix}0\\i\\0\\0\end{pmatrix}:-\gamma_{230},-\gamma_{3}$$

$$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}:-\gamma_{23},\gamma_{30} \qquad \begin{pmatrix}0\\0\\i\\0\end{pmatrix}:-\gamma_{310},-\gamma_{123}$$

$$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}:\gamma_{12},\gamma_{10} \qquad \begin{pmatrix}0\\0\\0\\i\end{pmatrix}:\gamma_{0},\gamma_{2}$$

rotate $\gamma_i \rightarrow \gamma_{i+1}$:

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}:1,-\gamma_{30} \qquad \begin{pmatrix}i\\0\\0\\0\end{pmatrix}:-\gamma_{230},\gamma_{2}$$

$$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}:\gamma_{12},\gamma_{0123} \qquad \begin{pmatrix}0\\i\\0\\0\end{pmatrix}:-\gamma_{310},-\gamma_{1}$$

$$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}:-\gamma_{31},\gamma_{10} \qquad \begin{pmatrix}0\\0\\i\\0\end{pmatrix}:-\gamma_{120},-\gamma_{123}$$

$$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}:\gamma_{23},\gamma_{20} \qquad \begin{pmatrix}0\\0\\0\\i\end{pmatrix}:\gamma_{0},\gamma_{3}$$

real Cl(3,1)

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} : 1, \gamma_2, \gamma_{30}, \gamma_{230} \qquad \begin{pmatrix}i\\0\\0\\0\end{pmatrix} :$$

$$\begin{pmatrix}0\\1\\0\\0\end{pmatrix} : -\gamma_1, -\gamma_{12}, \gamma_{310}, \gamma_{0123} \qquad \begin{pmatrix}0\\i\\0\\0\end{pmatrix} :$$

$$\begin{pmatrix}0\\0\\1\\0\end{pmatrix} : \gamma_{31}, \gamma_{123}, -\gamma_{10}, \gamma_{120} \qquad \begin{pmatrix}0\\0\\i\\0\end{pmatrix} :$$

$$\begin{pmatrix}0\\0\\0\\1\end{pmatrix} : \gamma_3, -\gamma_{23}, \gamma_0, -\gamma_{20} \qquad \begin{pmatrix}0\\0\\0\\i\end{pmatrix} :$$

alternative Majorana basis:

$$1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \gamma_{0123} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\gamma_0 = i \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_{123} = i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\gamma_1 = i \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad \gamma_{230} = i \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma_2 = i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma_{310} = i \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\gamma_3 = i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_{120} = i \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\gamma_{23} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_{10} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\gamma_{31} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \gamma_{20} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_{12} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \gamma_{30} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4}(1 - i\gamma_2)(1 - \gamma_{30}) = \frac{1}{4}(1 + i\gamma_{230})(1 - \gamma_{30})$$

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{4}(1 + i\gamma_2)(1 + \gamma_{30})$$

$$P = \frac{1}{4}(1 - i\gamma_2)(1 - \gamma_{30})$$

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} : 1, -\gamma_{30} \qquad \begin{pmatrix}i\\0\\0\\0\end{pmatrix} : -\gamma_{230}, \gamma_2$$

$$\begin{pmatrix}0\\1\\0\\0\end{pmatrix} : \gamma_{12}, \gamma_{0123} \qquad \begin{pmatrix}0\\i\\0\\0\end{pmatrix} : -\gamma_{310}, -\gamma_1$$

$$\begin{pmatrix}0\\0\\1\\0\end{pmatrix} : -\gamma_{31}, \gamma_{10} \qquad \begin{pmatrix}0\\0\\i\\0\end{pmatrix} : -\gamma_{120}, -\gamma_{123}$$

$$\begin{pmatrix}0\\0\\0\\1\end{pmatrix} : \gamma_{23}, \gamma_{20} \qquad \begin{pmatrix}0\\0\\0\\i\end{pmatrix} : \gamma_0, \gamma_3$$

$$P = \frac{1}{4}(1 + i\gamma_2)(1 + \gamma_{30})$$

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} : -\gamma_{23}, \gamma_{20} \qquad \begin{pmatrix}i\\0\\0\\0\end{pmatrix} : -\gamma_0, \gamma_3$$

$$\begin{pmatrix}0\\1\\0\\0\end{pmatrix} : -\gamma_{31}, -\gamma_{10} \qquad \begin{pmatrix}0\\i\\0\\0\end{pmatrix} : -\gamma_{120}, \gamma_{123}$$

$$\begin{pmatrix}0\\0\\1\\0\end{pmatrix} : -\gamma_{12}, \gamma_{0123} \qquad \begin{pmatrix}0\\0\\i\\0\end{pmatrix} : \gamma_{310}, -\gamma_1$$

$$\begin{pmatrix}0\\0\\0\\1\end{pmatrix} : 1, \gamma_{30} \qquad \begin{pmatrix}0\\0\\0\\i\end{pmatrix} : -\gamma_{230}, -\gamma_2$$

3,1 STA

$$1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \gamma_{0123} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

$$\gamma_0 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \qquad \gamma_{123} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_{230} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_{310} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \qquad \gamma_{120} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\gamma_{23} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix} \qquad \gamma_{10} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\gamma_{31} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad \gamma_{20} = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$\gamma_{12} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \qquad \gamma_{30} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$