This started out as a comment on chapter 3 and 4 of *C. Furey, Standard model physics from an algebra?* as well as *C. McKenzie, An Interpretation of Relativistic Spin Entanglement Using Geometric Algebra.* It has evolved quite a bit since then. NB: these notes are still a bit messy. Last updated March 27, 2025.

Complex Quaternions

Let ρ_1, ρ_2, ρ_3 be the quaternionic units and i the complex unit. This algebra $\mathbb{C} \otimes \mathbb{H}$ is isomorphic to $\mathrm{Cl}_{3,0}$, also called APS. We consider three involutions: complex conjugation e^*, which negates i, quaternionic conjugation $\tilde{e}$, which negates ρ_i and reverses order of multiplication, and hermitian conjugation $e^\dagger$, which does both. Complex conjugation corresponds to grade involution in $\mathrm{Cl}_{3,0}$, and hermitian conjugation to the reverse. Because in this algebra grade involution coincides with a parity transformation we can expect this operation to be important later on.

Elements of $\mathrm{Cl}_{3,0}$ can of course be written as matrices, but we have to be careful not to confuse the complex i in the matrix representation with the i of our $\mathbb{C} \otimes \mathbb{H}$ algebra because complex conjugation gives different results. For this reason we use j in the matrices to keep them distinct. Otherwise however σ_{123} behaves just like i. Hermitian conjugation is independent of representation: j-conjugation together with transposition has the same effect as i-conjugation and quaternionic conjugation. The matrix equivalent of quaternionic conjugation is adjugation (the diagonal elements are swapped, the off-diagonal elements are negated), and therefore the matrix equivalent of i-conjugation is everything combined: j-conjugation, transposition, adjugation. We will denote j-conjugation with e° and transposition with e^T.

The following table shows all the involutions, the ones marked with R also reverse the elements:

$\mathbb{C} \otimes \mathbb{H}$	$\mathrm{Cl}_{3,0}$	e^*	$\tilde{e}$	$e^\dagger$	e°	e^T
1	1	$+$	$+$	$+$	$+$	$+$
$i\rho_1$	σ_1	$-$	$-$	$+$	$+$	$+$
$i\rho_2$	σ_2	$-$	$-$	$+$	$-$	$-$
$i\rho_3$	σ_3	$-$	$-$	$+$	$+$	$+$
ρ_1	$-\sigma_{23}$	$+$	$-$	$-$	$-$	$+$
ρ_2	$-\sigma_{31}$	$+$	$-$	$-$	$+$	$-$
ρ_3	$-\sigma_{12}$	$+$	$-$	$-$	$-$	$+$
i	σ_{123}	$-$	$+$	$-$	$-$	$+$
			R	R		R

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_{123} = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad -\sigma_{23} = \begin{pmatrix} 0 & -j \\ -j & 0 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix} \qquad -\sigma_{31} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad -\sigma_{12} = \begin{pmatrix} -j & 0 \\ 0 & j \end{pmatrix}$$

Spinors

We will see that left and right Weyl spinors live in two different minimal left ideals. Their sum is a Dirac spinor and spans the whole algebra.

We start by defining a new basis:

$$S_{\uparrow\uparrow} = \tfrac{1}{2}(1 + i\rho_3) \qquad S_{\uparrow\downarrow} = \tfrac{1}{2}(i\rho_1 - \rho_2)$$
$$S_{\downarrow\uparrow} = \tfrac{1}{2}(i\rho_1 + \rho_2) \qquad S_{\downarrow\downarrow} = \tfrac{1}{2}(1 - i\rho_3)$$

Multiplying these elements is very intuitive: the product $S_{ab}S_{cd}$ (where $a, b, c, d \in \{\uparrow, \downarrow\}$) is $S_{ad}\delta_{bc}$, i.e. the middle two arrows cancel out if they align and annihilate otherwise. Thus $S_{\uparrow\uparrow}, S_{\downarrow\downarrow}$ are projectors and $S_{\uparrow\downarrow}, S_{\downarrow\uparrow}$ flip arrows. Note also that † swaps the two arrows. The following table shows the multiplication table as well as the effect of the conjugations:

	$S_{\uparrow\uparrow}$	$S_{\uparrow\downarrow}$	$S_{\downarrow\uparrow}$	$S_{\downarrow\downarrow}$	e^*	$\tilde{e}$	$e^\dagger$	e°	e^T
$S_{\uparrow\uparrow}$	$S_{\uparrow\uparrow}$	$S_{\uparrow\downarrow}$	0	0	$S_{\downarrow\downarrow}$	$S_{\downarrow\downarrow}$	$S_{\uparrow\uparrow}$	$S_{\uparrow\uparrow}$	$S_{\uparrow\uparrow}$
$S_{\uparrow\downarrow}$	0	0	$S_{\uparrow\uparrow}$	$S_{\uparrow\downarrow}$	$-S_{\downarrow\uparrow}$	$-S_{\uparrow\downarrow}$	$S_{\downarrow\uparrow}$	$S_{\uparrow\downarrow}$	$S_{\downarrow\uparrow}$
$S_{\downarrow\uparrow}$	$S_{\downarrow\uparrow}$	$S_{\downarrow\downarrow}$	0	0	$-S_{\uparrow\downarrow}$	$-S_{\downarrow\uparrow}$	$S_{\uparrow\downarrow}$	$S_{\downarrow\uparrow}$	$S_{\uparrow\downarrow}$
$S_{\downarrow\downarrow}$	0	0	$S_{\downarrow\uparrow}$	$S_{\downarrow\downarrow}$	$S_{\uparrow\uparrow}$	$S_{\uparrow\uparrow}$	$S_{\downarrow\downarrow}$	$S_{\downarrow\downarrow}$	$S_{\downarrow\downarrow}$

With this in hand we can construct the object $S_{\downarrow\uparrow} + S_{\uparrow\downarrow} = i\rho_1$ which flips any arrow, and the object $S_{\uparrow\uparrow} - S_{\downarrow\downarrow} = i\rho_3$, which tells us the direction of an arrow. $S_{\uparrow\uparrow} + S_{\downarrow\downarrow} = 1$ is trivial and $S_{\downarrow\uparrow} - S_{\uparrow\downarrow} = \rho_2 = (i\rho_1)(i\rho_3)$ is less interesting.

$$i\rho_1 S_{\uparrow x} = S_{\downarrow x} \qquad S_{x\uparrow} i\rho_1 = S_{x\downarrow}$$
$$i\rho_1 S_{\downarrow x} = S_{\uparrow x} \qquad S_{x\downarrow} i\rho_1 = S_{x\uparrow}$$

$$i\rho_3 S_{\uparrow x} = +S_{\uparrow x} \qquad S_{x\uparrow} i\rho_3 = +S_{x\uparrow}$$
$$i\rho_3 S_{\downarrow x} = -S_{\downarrow x} \qquad S_{x\downarrow} i\rho_3 = -S_{x\downarrow}$$

Note that it is not possible to flip the right arrow by left multiplication nor the left arrow by right multiplication. The $S_{\uparrow\uparrow}, S_{\downarrow\downarrow}$ projectors therefore partition the whole algebra into two minimal left ideals which we take to be the spaces of left and right Weyl spinors respectively. We can directly interpret the left arrow as indicating spin-up vs. spin-down states and the right arrow left- vs. right-handedness.

In matrix form the S-basis looks like this:

$$S_{\uparrow\uparrow} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad S_{\uparrow\downarrow} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$S_{\downarrow\uparrow} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad S_{\downarrow\downarrow} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Where left- vs. right-handed and spin-up vs. spin-down states have very suggestive spots in the matrix.

Because $\mathbb{C} \otimes \mathbb{H}$ elements can be gobbled up by the projectors we can also interpret left- and right-handed spinors as a real quaternion multiplied onto a projector. Of course all elements can also be expressed in terms of complex numbers and spin-up and -down states ($i\rho_1$ flips spin).

	$(1 + i\rho_3)/2$		$(1 - i\rho_3)/2$	
1	1	1	1	1
$i\rho_1$	ρ_2	$1(i\rho_1)$	$-\rho_2$	$1(i\rho_1)$
$i\rho_2$	$-\rho_1$	$i(i\rho_1)$	ρ_1	$-i(i\rho_1)$
$i\rho_3$	1	1	-1	-1
ρ_1	ρ_1	$-i(i\rho_1)$	ρ_1	$-i(i\rho_1)$
ρ_2	ρ_2	$1(i\rho_1)$	ρ_2	$-1(i\rho_1)$
ρ_3	ρ_3	$-i$	ρ_3	i
i	$-\rho_3$	i	ρ_3	i

From this we see that i is very closely related to ρ_3: multiplication with i is the same as multiplication with $\pm\rho_3$ at the projector, the sign depending on chirality. Likewise, multiplication by ρ_3 at the projector is the same as multiplication by $\pm i$.

Dirac algebra

Chiral representation

As we have seen, left multiplication with a $\mathbb{C}\otimes\mathbb{H}$ element cannot change chirality, but right multiplication with $i\rho_1$ does swap left and right. This is precisely what γ_0 does in the traditional approach, so we can identify γ_0 with right multiplication of $i\rho_1$: $\gamma_0 \equiv 1|i\rho_1$, such that $\gamma_0\psi = 1\psi i\rho_1 = \psi i\rho_1$.

Similarly we found that right-multiplication with $i\rho_3$ negates right-handed states, which is exactly what $-i\gamma_{0123} = -\gamma_5$ does. Therefore $-i\gamma_{0123} \equiv 1|i\rho_3$ and we recover the whole Dirac algebra as two-sided multiplication of $\mathbb{C}\otimes\mathbb{H}$ elements [1]:

$\mathrm{Cl}_{1,3}$	1	$i\rho_1$	$i\rho_2$	ρ_3
1	1	γ_0	$-\gamma_{123}$	$-\gamma_{0123}$
ρ_1	γ_{23}	γ_{023}	γ_1	$-\gamma_{10}$
ρ_2	γ_{31}	γ_{031}	γ_2	$-\gamma_{20}$
ρ_3	γ_{12}	γ_{012}	γ_3	$-\gamma_{30}$

E.g. $\gamma_1\gamma_{10} = (\rho_1|i\rho_2)(-\rho_1|\rho_3) = \rho_1(-\rho_1|\rho_3)i\rho_2 = (-\rho_1\rho_1)|(i\rho_3\rho_2) = -1|i\rho_1 = -\gamma_0$

Note that complex conjugation negates the odd elements, which is precisely what it does in the Majorana matrix representation. Therefore we find that complex conjugation corresponds to charge conjugation.

Dirac representation

If we take the $\frac{1}{2}(1 \pm i\rho_3)$ projectors instead to divide the spinor into positive and negative states rather than left- and right-handed ones we get the equivalent of the Dirac representation of the γ-matrices rather than the chiral one and the table becomes

$\mathrm{Cl}_{1,3}$	1	$i\rho_3$	$i\rho_2$	ρ_1
1	1	γ_0	γ_{123}	γ_{0123}
ρ_1	γ_{23}	γ_{023}	$-\gamma_1$	γ_{10}
ρ_2	γ_{31}	γ_{031}	$-\gamma_2$	γ_{20}
ρ_3	γ_{12}	γ_{012}	$-\gamma_3$	γ_{30}

These two are related by a reflected rotation of the right hand side: $R_D = -\sqrt{\rho_2}R_\chi\sqrt{-\rho_2}$.

Dirac's original matrices

To derive his famous equation, Dirac originally considered four matrices α_μ satisfying $\alpha_\mu\alpha_\nu + \alpha_\nu\alpha_\mu = 2\delta_{\mu\nu}$ $(\mu, \nu = 1, 2, 3, 4)$. He notes that the Pauli σ-matrices satisfy $\sigma_r\sigma_s + \sigma_s\sigma_r = 2\delta_{rs}$ $(r, s = 1, 2, 3)$ but that it is not possible to find a fourth such 2x2

[1] It is interesting to note that two-sided multiplication of the real quaternions gives $\mathrm{Cl}_{3,1}$, perhaps a hint that the $-+++$ metric is somewhat more fundamental.

$\mathrm{Cl}_{3,1}$	1	ρ_1	ρ_2	ρ_3
1	1	γ_0	γ_{123}	$-\gamma_{0123}$
ρ_1	$-\gamma_{23}$	$-\gamma_{023}$	γ_1	γ_{10}
ρ_2	$-\gamma_{31}$	$-\gamma_{031}$	γ_2	γ_{20}
ρ_3	$-\gamma_{12}$	$-\gamma_{012}$	γ_3	γ_{30}

matrix. Therefore he extends the σ-matrices to 4x4 matrices and finds a second set of matrices, ρ_r, which satisfy the same conditions. His ρ-matrices are obtained from interchanging the second and third rows and columns from his extended σ-matrices. To avoid confusion we call these two sets S and R instead:

$$S_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad S_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \qquad S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$R_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \qquad R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

He then finds $\alpha_r = R_1 S_r, \alpha_4 = R_3$ and defines a set of γ-matrices (which we will denote with Γ, again to avoid confusion): $\Gamma_r = R_2 S_r, \Gamma_4 = R_3$. These satisfy $\Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2\delta_{\mu\nu}$ just like the α's. Comparing with the now conventional Dirac γ-matrices we find the following equivalences:

$$\begin{aligned} S_1 &= i\gamma_{23} & R_1 &= i\gamma_{0123} \\ S_2 &= i\gamma_{31} & R_2 &= \gamma_{123} \\ S_3 &= i\gamma_{12} & R_3 &= \gamma_0 \\ \alpha_r &= \gamma_{0r} & \Gamma_r &= -i\gamma_r \\ \alpha_4 &= \gamma_0 & \Gamma_4 &= \gamma_0 \end{aligned}$$

The relationship between the S and R matrices becomes much clearer in the above light of left and right multiplication. Consider the two products of two matrices M and V, MV and VM:

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} aA + bC & cA + dC \\ aB + bD & cB + dD \end{pmatrix}$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} aA + cB & aC + cD \\ bA + dB & bC + dD \end{pmatrix}$$

Treating the elements of the matrix V as a vector v instead, we find the following for $M_L v$ and $M_R v$:

$$\begin{pmatrix} A & C & 0 & 0 \\ B & D & 0 & 0 \\ 0 & 0 & A & C \\ 0 & 0 & B & D \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} aA + bC \\ aB + bD \\ cA + dC \\ cB + dD \end{pmatrix}$$

$$\begin{pmatrix} A & 0 & B & 0 \\ 0 & A & 0 & B \\ C & 0 & D & 0 \\ 0 & C & 0 & D \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} aA + cB \\ bA + dB \\ aC + cD \\ bC + dD \end{pmatrix}$$

Thus we can identify Dirac's S and R matrices simply as the left and right multiplied σ-matrices interpreted as action on a vector. R_2 is in fact the negated version of right-multiplied σ_2 because right multiplication changes order: $iR_2 = R_3 R_1 \cong (1|\sigma_3)(1|\sigma_1) = (1|\sigma_{13}) = (1| - i\sigma_2)$.

Lorentz transformations

The ρ_i generate rotations, the $i\rho_i$ generate boosts. Complex conjugation being a parity transformation does not change rotations but flips boost direction. This means that if Λ is a left-handed Lorentz transformation, then Λ^* is the right-handed counterpart.

Transforming a full Dirac spinor $\psi = \psi_L + \psi_R = \psi S_{\uparrow\uparrow} + \psi S_{\downarrow\downarrow}$ with a Lorentz transformation of the form $\Lambda = e^{\rho_k c^k}$ ($c^k \in \mathbb{C}$) then takes the following form:

$$\psi \to \Lambda\psi_L + \Lambda^*\psi_R = \Lambda\psi S_{\uparrow\uparrow} + \Lambda^*\psi S_{\downarrow\downarrow} = \Lambda\psi S_{\uparrow\uparrow} + \Lambda^*\psi S^*_{\uparrow\uparrow}$$

Having to transform ψ_L and ψ_R explicitly is a bit tedious however and the reason is explict handling of i. But as we have seen i can be interpreted as a non-chiral version of right-multplied ρ_3. Therefore the replacement $i\rho_i|1 \to \rho_i|\rho_3$ automatically gives us boosts respecting chirality, which are exactly the γ_{i0} introduced above. So Lorentz transformations can be written more succinctly like this:

$$\begin{aligned}
\gamma_{jk}\psi &= (\rho_i|1)\psi = \rho_i\psi_L + \rho_i\psi_R \\
&= \rho_i\psi_L + \rho_i^*\psi_R \\
\gamma_{i0}\psi &= -(\rho_i|\rho_3)\psi = (i\rho_i|i\rho_3)\psi \\
&= i\rho_i\psi_L i\rho_3 + i\rho_i\psi_R i\rho_3 \\
&= i\rho_i\psi_L - i\rho_i\psi_R \\
&= i\rho_i\psi_L + (i\rho_i)^*\psi_R
\end{aligned}$$

When treating the traditional chiral column spinor as a minimal left ideal of the full Dirac algebra the projector that generates this ideal is $P = \frac{1}{4}(1 + i\gamma_{12})(1 + \gamma_{30})$ and represents the left up state. Our equivalent of this state is the $\frac{1}{2}(1+i\rho_3)$ projector. Note that in both cases $i\gamma_{12} \simeq (i\rho_3|1)$, $\gamma_{30} \simeq (i\rho_3|i\rho_3)$ and $-i\gamma_{0123} \simeq (1|i\rho_3)$ are swallowed by the projector(s) and do not affect the state at all.

In the Dirac basis the equivalent elements are $i\gamma_{12} \simeq (i\rho_3|1)$, $i\gamma_{012} \simeq (i\rho_3|i\rho_3)$, $\gamma_0 \simeq (1|i\rho_3)$ and $P = \frac{1}{4}(1 + i\gamma_{12})(1 + \gamma_0)$.

The geometric meaning of i, which is usually considered to be somewhat mysterious, now becomes rather clear. Let's see what happens when we multiply a spinor ψ by a phase $e^{i\theta}$:

$$\begin{aligned}
e^{i\theta}\psi &= e^{i\theta}\psi\tfrac{1}{2}(1 + i\rho_3) + e^{i\theta}\psi\tfrac{1}{2}(1 - i\rho_3) \\
&= \psi e^{i\theta}\tfrac{1}{2}(1 + i\rho_3) + \psi e^{i\theta}\tfrac{1}{2}(1 - i\rho_3) \\
&= \psi e^{-\rho_3\theta}\tfrac{1}{2}(1 + i\rho_3) + \psi e^{\rho_3\theta}\tfrac{1}{2}(1 - i\rho_3) \\
&= \psi e^{-\gamma_{12}\theta}\tfrac{1}{2}(1 + i\rho_3) + \psi e^{\gamma_{12}\theta}\tfrac{1}{2}(1 - i\rho_3)
\end{aligned}$$

This is simply a rotation in the *local* γ_{12} axis, which is the axis of spin.

Another meaning concerns dualisation of $\mathrm{Spin}^+(1,3)$ generators, which we have already seen above. In the STA/Dirac algebra, rotation and boost generators are related by dualisation through the pseudoscalar $I = \gamma_{0123}$ like $\gamma_{i0} = I\gamma_{jk}\epsilon_{ijk}$. But as $I = -(1|\rho_3)$ becomes $\pm i$ at the projector and is therefore the chiral version of i, we see that $i\gamma_{jk}$ and $I\gamma_{jk}$ are indeed just chirality-ignoring and chirality-respecting versions of the boost generators. Similarly $iI\gamma_{jk} = i\gamma_{i0}$ are rotation generators which affect both chiralities in opposite sense. This can be used to transform only left-handed spinors with $e^{(1-iI)\gamma_{i0}\theta/2}$ and $e^{(1-iI)\gamma_{jk}\theta/2}$, and only right handed ones with $e^{(1+iI)\gamma_{i0}\theta/2}$ and $e^{(1+iI)\gamma_{jk}\theta/2}$.

Inner product

We have $\widetilde{\Lambda} = \Lambda^{-1}$. So the following is Lorentz-invariant:

$$\begin{aligned}
\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L &\to \psi_L^\dagger \Lambda^\dagger \Lambda^* \psi_R + \psi_R^\dagger \widetilde{\Lambda}\,\Lambda \psi_L \\
&= \psi_L^\dagger (\widetilde{\Lambda}\,\Lambda)^* \psi_R + \psi_R^\dagger \widetilde{\Lambda}\,\Lambda \psi_L \\
&= \psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L
\end{aligned}$$

To be more explicit let $\psi_L = aS_{\uparrow\uparrow} + bS_{\downarrow\uparrow}$ and $\psi_R = cS_{\uparrow\downarrow} + dS_{\uparrow\uparrow}$.

$$\begin{aligned}
\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L &= a^* c S_{\uparrow\downarrow} + b^* d S_{\uparrow\downarrow} + c^* a S_{\downarrow\uparrow} + d^* b S_{\downarrow\uparrow} \\
&= (a^* c + b^* d) S_{\uparrow\downarrow} + (c^* a + d^* b) S_{\downarrow\uparrow}
\end{aligned}$$

These are the coefficients familiar from the the matrix-vector formalism, but unlike with column/row spinors this product is not a scalar yet. That can be achieved by use of $i\rho_1$:

$$\begin{aligned}
& i\rho_1(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L) + (\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L) i\rho_1 \\
&= (a^* c + b^* d) S_{\downarrow\downarrow} + (c^* a + d^* b) S_{\uparrow\uparrow} + (a^* c + b^* d) S_{\uparrow\uparrow} + (c^* a + d^* b) S_{\downarrow\downarrow} \\
&= a^* c + b^* d + c^* a + d^* b
\end{aligned}$$

Four types of spinors

$$\begin{array}{cccccc}
\mathrm{M} & M^* & \widetilde{M} & M^\dagger & M^\circ & M^T \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix} &
\begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix} &
\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} &
\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} &
\begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} &
\begin{pmatrix} a & c \\ b & d \end{pmatrix}
\end{array}$$

$$\begin{array}{cccc}
\text{left} & \text{right dual} & \text{left dual} & \text{right} \\
\begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & -\psi_2^* \\ 0 & \psi_1^* \end{pmatrix} &
\begin{pmatrix} 0 & 0 \\ -\psi_2 & \psi_1 \end{pmatrix} &
\begin{pmatrix} \psi_1^* & \psi_2^* \\ 0 & 0 \end{pmatrix}
\end{array}$$

$$\begin{array}{cccc}
\text{left} & \text{right } \epsilon & \text{right dual} & \text{left dual } \epsilon \\
\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} &
\begin{pmatrix} -\psi_2^* & \psi_1^* \end{pmatrix} &
\begin{pmatrix} \psi_1^* \\ \psi_2^* \end{pmatrix} &
\begin{pmatrix} -\psi_2 & \psi_1 \end{pmatrix}
\end{array}$$

In the following ψ_L &c. are taken to be raw column spinors while φ, ψ are full matrices (or complex quaternions). This is just to express inner products between individual Weyl spinors more succinctly. The goal is to derive an expression for the inner product of two Dirac spinors. We start with the obvious $\varphi^\dagger \psi$:

$$\begin{aligned}
\varphi^\dagger \psi &= \begin{pmatrix} \varphi_L^\dagger \\ \varphi_R^\dagger \end{pmatrix} \begin{pmatrix} \psi_L & \psi_R \end{pmatrix} \\
&= \begin{pmatrix} \varphi_L^\dagger \psi_L & \varphi_L^\dagger \psi_R \\ \varphi_R^\dagger \psi_L & \varphi_R^\dagger \psi_R \end{pmatrix}
\end{aligned}$$

The Lorentz invariant terms are on the anti-diagonal. We can multiply (on any side) with $i\rho_1$, which is the same as sandwiching γ_0:

$$\varphi^\dagger \gamma_0 \psi = \varphi^\dagger \psi i\rho_1 = \begin{pmatrix} \varphi_L^\dagger \psi_L & \varphi_L^\dagger \psi_R \\ \varphi_R^\dagger \psi_L & \varphi_R^\dagger \psi_R \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \varphi_L^\dagger \psi_R & \varphi_L^\dagger \psi_L \\ \varphi_R^\dagger \psi_R & \varphi_R^\dagger \psi_L \end{pmatrix}$$

The trace of this is the familiar inner product, however we can do better by adding the quaternion conjugate:

$$\begin{aligned}
\varphi^\dagger\psi i\rho_1 + \widetilde{\varphi^\dagger\psi i\rho_1} &= \varphi^\dagger\psi i\rho_1 - i\rho_1\,\widetilde{\psi}\,\varphi^* \\
&= \begin{pmatrix} \varphi_L^\dagger\psi_R & \varphi_L^\dagger\psi_L \\ \varphi_R^\dagger\psi_R & \varphi_R^\dagger\psi_L \end{pmatrix} + \begin{pmatrix} \varphi_R^\dagger\psi_L & -\varphi_L^\dagger\psi_L \\ -\varphi_R^\dagger\psi_R & \varphi_L^\dagger\psi_R \end{pmatrix} \\
&= \begin{pmatrix} \varphi_L^\dagger\psi_R + \varphi_R^\dagger\psi_L & 0 \\ 0 & \varphi_L^\dagger\psi_R + \varphi_R^\dagger\psi_L \end{pmatrix} \\
&= \varphi_L^\dagger\psi_R + \varphi_R^\dagger\psi_L
\end{aligned}$$

In the Dirac representation γ_0 is right multiplication with $i\rho_3$ instead and so there we get:

$$\varphi^\dagger\gamma_0\psi = \varphi^\dagger\psi i\rho_3 = \begin{pmatrix} \varphi_+^\dagger\psi_+ & \varphi_+^\dagger\psi_- \\ \varphi_-^\dagger\psi_+ & \varphi_-^\dagger\psi_- \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \varphi_+^\dagger\psi_+ & -\varphi_+^\dagger\psi_- \\ \varphi_-^\dagger\psi_+ & -\varphi_-^\dagger\psi_- \end{pmatrix}$$

And we again take the trace to get the familiar expression for the inner product.

TODO: clarify. not taking the scalar part seems to break associativity. the following is ambiguous depending on whether γ_0 associates to the left or right:

$$\begin{aligned}
&\varphi^\dagger\gamma_0\psi = \varphi^\dagger\psi i\rho_1 \\
&\varphi^\dagger\gamma_0\psi = \varphi^\dagger\gamma_0^\dagger\psi = (\gamma_0\varphi)^\dagger\psi = (\varphi i\rho_1)^\dagger\psi = i\rho_1\varphi^\dagger\psi
\end{aligned}$$

Vectors from Spinor products

Algebraic product

Just an idea, is this useful for anything?

$$\begin{aligned}
\Lambda i\rho_3\Lambda^\dagger &= \Lambda\tfrac{1}{2}(1+i\rho_3)\Lambda^\dagger - \Lambda\tfrac{1}{2}(1-i\rho_3)\Lambda^\dagger \\
&= \left[\Lambda\tfrac{1}{2}(1+i\rho_3)\right]\left[\tfrac{1}{2}(1+i\rho_3)\Lambda^\dagger\right] - \left[\Lambda\tfrac{1}{2}(1-i\rho_3)\right]\left[\tfrac{1}{2}(1-i\rho_3)\Lambda^\dagger\right] \\
&= \left[\Lambda\tfrac{1}{2}(1+i\rho_3)\right]\left[\Lambda\tfrac{1}{2}(1+i\rho_3)\right]^\dagger - \left[\Lambda\tfrac{1}{2}(1-i\rho_3)\right]\left[\Lambda\tfrac{1}{2}(1-i\rho_3)\right]^\dagger
\end{aligned}$$

For a null-vector this is nice:

$$\begin{aligned}
\Lambda\tfrac{1}{2}(1+i\rho_3)\Lambda^\dagger &= \left[\Lambda\tfrac{1}{2}(1+i\rho_3)\right]\left[\Lambda\tfrac{1}{2}(1+i\rho_3)\right]^\dagger \\
&= \psi_L\psi_L^\dagger
\end{aligned}$$

Tensor product

We will show how the tensor product of two Pauli spinors results in an object with spin-0 and spin-1 components. We begin by defining product up and down states for two spinors:

$$|{\uparrow\uparrow}\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad |{\uparrow\downarrow}\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad |{\downarrow\uparrow}\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad |{\downarrow\downarrow}\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

The spin matrices for the first and second spinor then are:

$$S_z^1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad S_y^1 = \begin{pmatrix} & & -i & \\ & & & -i \\ i & & & \\ & i & & \end{pmatrix} \qquad S_x^1 = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix}$$

$$S_z^2 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \qquad S_y^2 = \begin{pmatrix} & -i & & \\ i & & & \\ & & & -i \\ & & i & \end{pmatrix} \qquad S_x^2 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}$$

And the total spin matrices are given by their sum:

$$S_z = \begin{pmatrix} 2 & & & \\ & 0 & & \\ & & 0 & \\ & & & -2 \end{pmatrix} \qquad S_y = \begin{pmatrix} & -i & -i & \\ i & & & -i \\ i & & & -i \\ & i & i & \end{pmatrix} \qquad S_x = \begin{pmatrix} & 1 & 1 & \\ 1 & & & 1 \\ 1 & & & 1 \\ & 1 & 1 & \end{pmatrix}$$

A change of basis now gives us three spin-1 components and one spin-0 component:

$$|z_+\rangle = |\uparrow\uparrow\rangle \qquad |z_0\rangle = \tfrac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle) \qquad |z_-\rangle = |\downarrow\downarrow\rangle$$
$$|0\rangle = \tfrac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)$$

We now repeat this construction but using a $\mathbb{C} \otimes \mathbb{H}$ (or matrix) basis instead of a column vector basis. The basis elements we have already seen in the context of Dirac spinors, but here they have different meaning. Instead of a left-right split we have a second up-down split:

$$|\uparrow\uparrow\rangle = \tfrac{1}{2}(1 + i\rho_3) \cong \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad |\uparrow\downarrow\rangle = \tfrac{1}{2}(i\rho_1 - \rho_2) \cong \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$|\downarrow\uparrow\rangle = \tfrac{1}{2}(i\rho_1 + \rho_2) \cong \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad |\downarrow\downarrow\rangle = \tfrac{1}{2}(1 - i\rho_3) \cong \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The spin operators now act from the left and right respectively, and we find:

$$\begin{aligned} S_z^1 &= i\rho_3|1 & S_y^1 &= i\rho_2|1 & S_x^1 &= i\rho_1|1 \\ S_z^2 &= 1|i\rho_3 & S_y^2 &= -1|i\rho_2 & S_x^2 &= 1|i\rho_1 \\ S_z &= (i\rho_3|1) + (1|i\rho_3) & S_y &= (i\rho_2|1) - (1|i\rho_2) & S_x &= (i\rho_1|1) + (1|i\rho_1) \end{aligned}$$

And the other basis:

$$|z_+\rangle = \tfrac{1}{2}(1 + i\rho_3) \qquad |z_0\rangle = \tfrac{1}{\sqrt{2}} i\rho_1 \qquad |z_-\rangle = \tfrac{1}{2}(1 - i\rho_3)$$
$$|0\rangle = \tfrac{1}{\sqrt{2}}\rho_2$$

Clearly ρ_2 is a bit special, so we write the above in a somewhat more suggestive way:

$$|z_+\rangle = \tfrac{1}{2}(i\rho_2 + \rho_1)(i\rho_2) \qquad |z_0\rangle = \tfrac{-1}{\sqrt{2}}\rho_3(i\rho_2) \qquad |z_-\rangle = \tfrac{1}{2}(i\rho_2 - \rho_1)(i\rho_2)$$
$$|0\rangle = \tfrac{-i}{\sqrt{2}}(i\rho_2)$$

Sandwiching the S_i with $(1|i\rho_2)$ gives us the familiar SO(3) generators L_i:

$$\begin{aligned} L_z &= (1|i\rho_2)S_z(1|i\rho_2) = (i\rho_3|1) - (1|i\rho_3) \\ L_y &= (1|i\rho_2)S_y(1|i\rho_2) = (i\rho_2|1) - (1|i\rho_2) \\ L_x &= (1|i\rho_2)S_x(1|i\rho_2) = (i\rho_1|1) - (1|i\rho_1) \end{aligned}$$

It is now obvious that a (complex) quaternion splits into the scalar and vector part under L_i, transforming as spin-0 and spin-1 respectively.

Quaternions as Weyl spinors

As we have seen the pure quaternions generate Cl$(3,1)$ by two-sided multiplication. As it turns out the sandwiched quaternion transforms as a Weyl spinor. We can use this to build the Pauli algebra purely out of quaternions and use it to transform our spinors. Because left and right multiplication commute, right-multiplied ρ_3 commutes with any left-multiplied quaternion and thus generates the complex numbers. This construction therefore is how we arrive at $\mathbb{C} \otimes \mathbb{H}$.

$$\begin{aligned} 1 &= (1|1) & \sigma_1\sigma_2\sigma_3 = i &\cong (1|\widetilde{\rho_3}) \\ \sigma_1 = i\rho_1 &\cong (\rho_1|\widetilde{\rho_3}) & -\sigma_{23} &= (\rho_1|1) \\ \sigma_2 = i\rho_2 &\cong (\rho_2|\widetilde{\rho_3}) & -\sigma_{31} &= (\rho_2|1) \\ \sigma_3 = i\rho_3 &\cong (\rho_3|\widetilde{\rho_3}) & -\sigma_{12} &= (\rho_3|1) \end{aligned}$$

One can easily confirm that a column spinor and its complex conjugate can then be expressed as follows:

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}a+ib\\ c+id\end{pmatrix} \cong (a+\widetilde{\rho_3}\,b) + \rho_2(c+\widetilde{\rho_3}\,d) = \psi_1 + \rho_2\psi_2$$
$$\psi^* \cong -\rho_2\psi\rho_2$$

For the inner product of two spinors we find the following expression in the row-column formalism:

$$\psi^\dagger\varphi = \begin{pmatrix}\psi_1^* & \psi_2^*\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \psi_1^*\varphi_1 + \psi_2^*\varphi_2$$

To find the quaternion equivalent we first calculate $\widetilde{\psi}\,\varphi$:

$$\begin{aligned} \widetilde{\psi}\,\varphi &= (\widetilde{\psi_1} - \widetilde{\psi_2}\,\rho_2)(\varphi_1 + \rho_2\varphi_2) \\ &= (\widetilde{\psi_1} - \rho_2\psi_2)(\varphi_1 + \widetilde{\varphi_2}\,\rho_2) \\ &= \widetilde{\psi_1}\,\varphi_1 - \rho_2\psi_2\,\widetilde{\varphi_2}\,\rho_2 + \widetilde{\psi_1}\,\widetilde{\varphi_2}\,\rho_2 - \rho_2\psi_2\varphi_1 \\ &= (\widetilde{\psi_1}\,\varphi_1 + \widetilde{\psi_2}\,\varphi_2) + \rho_2(\psi_1\varphi_2 - \psi_2\varphi_1) \end{aligned}$$

It is the first part here that corresponds to the conventional inner product. We can get rid of the second part by sandwiching with ρ_3, which can also be interpreted as a projection onto ρ_3.

$$\begin{aligned} \psi^\dagger\varphi &\cong \tfrac{1}{2}(\widetilde{\psi}\,\varphi - \rho_3\,\widetilde{\psi}\,\varphi\rho_3) \\ &= -\tfrac{1}{2}(\widetilde{\psi}\,\varphi\rho_3 + \rho_3\,\widetilde{\psi}\,\varphi)\rho_3 \\ &= -[(\widetilde{\psi}\,\varphi)\cdot\rho_3]\rho_3 \\ &= \tfrac{1}{2}(1+i\rho_3)(\widetilde{\psi}\,\varphi) \end{aligned}$$

In the common case of sandwiching the hermitian elements σ_i between the same spinor this simplifies:

$$\begin{aligned} \psi^\dagger\sigma_i\psi &\cong \tfrac{1}{2}(\widetilde{\psi}\,\rho_i\psi\,\widetilde{\rho_3} - \rho_3\,\widetilde{\psi}\,\rho_i\psi\,\widetilde{\rho_3}\,\rho_3) \\ &= \tfrac{1}{2}(\widetilde{\psi}\,\rho_i\psi\,\widetilde{\rho_3} + \widetilde{\rho_3}\,\widetilde{\psi}\,\rho_i\psi) \\ &= (\widetilde{\psi}\,\rho_i\psi)\cdot\widetilde{\rho_3} \end{aligned}$$

Clifford vs Cayley-Dickson

Where does the Cl(3, 1) algebra come from and why is the even subalgebra specifically picked out? A possible answer might lie in the intersection of Clifford algebras and Cayley-Dickson algebras. Remarkably, multiplication in both types of algebras of up to two dimensions are completely identical, however they both come equipped with distinct involutions.

The Cayley-Dickson construction is canonically defined recursively by a conjugate $\overline{(a, b)} = (\overline{a}, -b)$ and a multiplication $(a, b)(c, d) = (ac - \overline{d}\, b, da + b\,\overline{c})$. One can easily check that $\overline{ab} = \overline{b}\,\overline{a}$. Alternatively, to take a more minimal approach, it is enough to only define multiplication, this time between an element and a conjugate element: $(a, b)\,\overline{(c, d)} = (a\,\overline{c} + \overline{d}\, b, -da + bc)$. We then get conjugation for free in terms of a product with the identity: $\overline{a} = (1, 0)\,\overline{a}$.

A similar definition allows us to integrate the reverse of Clifford algebras into the product, $a\,\tilde{b}$. The reverse is defined recursively $\widetilde{ab} = \tilde{b}\,\tilde{a}$ with basis vectors being unaffected ($\widetilde{\gamma_i} = \gamma_i$).

Two elements of both algebras can be identified by the binary code that indexes them. The respective involutions flip signs as follows (note that ∼ here now refers to reversion in Cl(0, 2), unlike earlier):

$\mathbb{H}$	$\mathrm{Cl}_{0,2}$	$\overline{e}$	$\tilde{e}$
1	1	+	+
ρ_1	γ_1	−	+
ρ_2	γ_2	−	+
ρ_3	γ_{12}	−	−

Defining multiplication with the involution included now results in larger left-multiplication algebras than we would have gotten otherwise. In particular, chaining left-multiplications now allows us to multiply on the right as well:

$$a\,\overline{(b\,\overline{\psi})} = a\psi\,\overline{b}$$
$$a\,\widetilde{(b\,\widetilde{\psi})} = a\psi\,\tilde{b}$$

Requiring these two expressions to be exactly equal restricts the right element b (but not the left) to lie in the span of $\{1, \rho_3\}$. The intersection of an even number of left-multiplications in both algebras therefore gives us exactly $\mathrm{Cl}^+(3, 1)$, or $\mathbb{C} \otimes \mathbb{H}$.

The rotation and boost generators can now be expressed as:

$$\rho_i\,\widetilde{(1\,\widetilde{\psi})} = \rho_i\psi$$
$$\rho_i\,\widetilde{(\rho_3\,\widetilde{\psi})} = \rho_i\psi\,\widetilde{\rho_3}$$

Isomorphisms

$\mathbb{C}\otimes\mathbb{H}$	$\mathrm{Cl}_{3,0}$	$\mathrm{Cl}_{1,2}$	$\mathrm{Cl}^+_{3,1}$	$\mathrm{Cl}^+_{1,3}$	$\mathbb{C}\mathrm{l}_{2,0}$	$\mathbb{C}\mathrm{l}_{1,1}$	$\mathbb{C}\mathrm{l}^+_{2,1}$	$\mathbb{C}\mathrm{l}^+_{1,2}$	$\mathbb{C}\mathrm{l}_{0,2}$	$\mathbb{C}\mathrm{l}^+_{3,0}$	$\mathbb{C}\mathrm{l}^+_{0,3}$
$i\rho_1$	e_1	γ_{10}	e_{10}	γ_{10}	e_1	e_+	e_{10}	γ_{10}	$i\gamma_1$	$-ie_{23}$	$i\gamma_{23}$
$i\rho_2$	e_2	γ_{20}	e_{20}	γ_{20}	e_2	ie_-	e_{20}	γ_{20}	$i\gamma_2$	$-ie_{31}$	$i\gamma_{31}$
$i\rho_3$	e_3	γ_0	e_{30}	γ_{30}	$-ie_{12}$	e_{+-}	$-ie_{12}$	$i\gamma_{12}$	$i\gamma_{12}$	$-ie_{12}$	$i\gamma_{12}$
$-\rho_1$	e_{23}	γ_2	e_{23}	$-\gamma_{23}$	ie_1	ie_+	ie_{10}	$i\gamma_{10}$	$-\gamma_1$	e_{23}	$-\gamma_{23}$
$-\rho_2$	e_{31}	$-\gamma_1$	e_{31}	$-\gamma_{31}$	ie_2	$-e_-$	ie_{20}	$i\gamma_{20}$	$-\gamma_2$	e_{31}	$-\gamma_{31}$
$-\rho_3$	e_{12}	$-\gamma_{12}$	e_{12}	$-\gamma_{12}$	e_{12}	ie_{+-}	e_{12}	$-\gamma_{12}$	$-\gamma_{12}$	e_{12}	$-\gamma_{12}$
i	e_{123}	$-\gamma_{012}$	$-e_{0123}$	γ_{0123}	i	i	i	i	i	i	i

Octonion product

The octonions as well as $\mathbb{C} \otimes \mathbb{H}$ can be thought of as a pair of quaternions, but with different rules of multiplication. If we call the real and imaginary parts of a complex quaternion $q_R = \frac{1}{2}(q + q^*)$ and $q_I = \frac{1}{2}(q - q^*)$ we can express the octonion product like this, which is just the rule of Cayley-Dickson multiplication in slightly different language (also see Lasenby):

$$q \star p := q_R p_R + \widetilde{p_I}\, q_I + p_I q_R + q_I\, \widetilde{p_R}$$