Pauli Spinors as Quaternions

We start with the subalgebra of STA that includes rotations (= quaternions), i.e. Cl(0,3). Traditionally a Pauli spinor is broken up into a up and a down component, each multiplied with a complex number. To do this in GA we pick a reference spin-axis (γ_{12}) and build a projector from it:

$$P := \frac{1}{2}(1 + i\gamma_{12})$$

Note the important eigenvalue relationship, which will allow us to interpret the complex i simply as multiplication by $-\gamma_{12}$ (or perhaps better $\tilde{\gamma_{12}}$) *at the projector*:

$$\begin{aligned} i\gamma_{12}P &= P \\ -\gamma_{12}P &= iP \end{aligned}$$

Our general Pauli spinor is then any quaternion multiplied onto P on the left:

$$\begin{aligned} \psi &= (w + z\gamma_{12} + x\gamma_{23} + y\gamma_{31})P \\ &= (w + z\gamma_{12})P + (y - x\gamma_{12})\gamma_{31}P \\ &= (w + z\gamma_{12})P + \gamma_{31}(y + x\gamma_{12})P \\ &= (w - zi)P + \gamma_{31}(y - xi)P \\ &= (w - zi)P + (y - xi)\gamma_{31}P \\ &= (w - zi)\left|\uparrow\right\rangle + (y - xi)\left|\downarrow\right\rangle \end{aligned}$$

with $\left|\uparrow\right\rangle := 1P$ and $\left|\downarrow\right\rangle := \gamma_{31}P$. Note that for any $q \in \mathrm{Cl}^+(0,3)$ we can express right-multiplication by $\tilde{\gamma_{12}}$ as multiplication with the (commutative!) complex i:

$$iqP = qiP = q\tilde{\gamma_{12}}P$$

As Hestenes observed we can alternatively introduce i as an *operator* that is defined to act this way, which allows us to get rid of the projector. In that case the Pauli spinor is truly nothing more than a quaternion. In any case the complex i has been turned into mere "syntax", devoid of any additional geometric meaning. It does remain useful however, because we have picked γ_{12} as our reference spin axis, to which we may want to refer to on either side. This also allows us to write a sandwich product using only left-multiplication: $i\gamma_{12}\psi = \gamma_{12}\psi i = \gamma_{12}\psi\tilde{\gamma_{12}}$.

This latter fact is what allows us to define the σ spin operators. We will start with the z-direction and derive the general case by rotating the world. We require $\sigma_z\left|\uparrow\right\rangle = +\left|\uparrow\right\rangle$ and $\sigma_z\left|\downarrow\right\rangle = -\left|\downarrow\right\rangle$. But this is precisely what $i\gamma_{12}$ does! I.e. it separates the component of ψ which commutes with the spin axis (γ_{12}) from the one which anti-commutes with it. We have:

$$\begin{aligned} \sigma_z &= i\gamma_{12} \\ \sigma_z\psi &= i\gamma_{12}(w + z\gamma_{12} + x\gamma_{23} + y\gamma_{31})P \\ &= \gamma_{12}(w + z\gamma_{12} + x\gamma_{23} + y\gamma_{31})\tilde{\gamma_{12}}P \\ &= (w + z\gamma_{12} - x\gamma_{23} - y\gamma_{31})P \\ &= (w - zi)\left|\uparrow\right\rangle - (y - xi)\left|\downarrow\right\rangle \end{aligned}$$

To deal with arbitrary spin axes we cannot simply rotate this whole expression because that would also rotate our reference axis i. Instead we leave i fixed but rotate the rest of the world. Let our arbirary spin axis be called s and its corresponding spin operator

σ_s. Then a rotor taking γ_{12} to s is given by $R = \sqrt{-s\gamma_{12}} = \frac{1}{\sqrt{2}}(1 - s\gamma_{12})$. We see that the following transformations give us the correct eigenvalues of σ_s:

$$\begin{aligned}\psi_z &\to \psi_s = R\psi_z \\ \sigma_z &\to \sigma_s = R\sigma_z\tilde{R} = iR\gamma_{12}\tilde{R}\end{aligned}$$

$$\begin{aligned}\sigma_s\psi_s^{\pm} &= (iR\gamma_{12}\tilde{R})(R\psi_z^{\pm}) \\ &= iR\gamma_{12}\psi_z^{\pm} = R(i\gamma_{12}\psi_z^{\pm}) \\ &= \pm R\psi_z^{\pm} = \pm\psi_s^{\pm}\end{aligned}$$

In particular, for the x and y axes we find:

$$\begin{aligned}\sigma_x &= \sqrt{-\gamma_{23}\gamma_{12}}(i\gamma_{12})\sqrt{-\gamma_{12}\gamma_{23}} \\ &= i\sqrt{-\gamma_{23}\gamma_{12}}\sqrt{-\gamma_{23}\gamma_{12}}\gamma_{12} \\ &= i(-\gamma_{23}\gamma_{12})\gamma_{12} \\ &= i\gamma_{23}\end{aligned}$$

$$\begin{aligned}\sigma_y &= \sqrt{-\gamma_{31}\gamma_{12}}(i\gamma_{12})\sqrt{-\gamma_{12}\gamma_{31}} \\ &= i\gamma_{31}\end{aligned}$$

$$\begin{aligned}|{\uparrow_x}\rangle &= \sqrt{-\gamma_{23}\gamma_{12}}\,|{\uparrow}\rangle = \frac{1}{\sqrt{2}}(1+\gamma_{31})\,|{\uparrow}\rangle \\ &= \frac{1}{\sqrt{2}}(|{\uparrow}\rangle + |{\downarrow}\rangle)\end{aligned}$$

$$\begin{aligned}|{\downarrow_x}\rangle &= \sqrt{-\gamma_{23}\gamma_{12}}\,|{\downarrow}\rangle = \frac{1}{\sqrt{2}}(1+\gamma_{31})\,|{\downarrow}\rangle \\ &= \frac{1}{\sqrt{2}}(-\,|{\uparrow}\rangle + |{\downarrow}\rangle) = -\frac{1}{\sqrt{2}}(|{\uparrow}\rangle - |{\downarrow}\rangle)\end{aligned}$$

$$\begin{aligned}|{\uparrow_y}\rangle &= \sqrt{-\gamma_{31}\gamma_{12}}\,|{\uparrow}\rangle = \frac{1}{\sqrt{2}}(1+\gamma_{12}\gamma_{31})\,|{\uparrow}\rangle \\ &= \frac{1}{\sqrt{2}}(|{\uparrow}\rangle + \gamma_{12}\,|{\downarrow}\rangle) = \frac{1}{\sqrt{2}}(|{\uparrow}\rangle + i\,|{\downarrow}\rangle)\end{aligned}$$

$$\begin{aligned}|{\downarrow_y}\rangle &= \sqrt{-\gamma_{31}\gamma_{12}}\,|{\downarrow}\rangle = \frac{1}{\sqrt{2}}(1+\gamma_{12}\gamma_{31})\,|{\downarrow}\rangle \\ &= \frac{1}{\sqrt{2}}(-\gamma_{12}\,|{\uparrow}\rangle + |{\downarrow}\rangle) = i\frac{1}{\sqrt{2}}(|{\uparrow}\rangle - i\,|{\downarrow}\rangle)\end{aligned}$$

Where the two down states picked up inconsequential phases.

It is easy to check that σ_i obey the usual commutation relations:

$$\begin{aligned}\sigma_x\sigma_y &= (i\gamma_{23})(i\gamma_{31}) = -\gamma_{12} = i\sigma_z \\ \sigma_y\sigma_z &= (i\gamma_{31})(i\gamma_{12}) = -\gamma_{23} = i\sigma_y \\ \sigma_z\sigma_x &= (i\gamma_{12})(i\gamma_{23}) = -\gamma_{31} = i\sigma_x\end{aligned}$$

Stern-Gerlach experiment

We can now investigate what happens in the SG experiment. The interaction between the magnetic field and the spin of the electron in the Pauli-Hamiltonian is usually given as $\boldsymbol{\sigma} \cdot \mathbf{B} \left|\psi\right\rangle$ (we ignore various physical constants). Since the magnetic field is best thought of as a bivector, we can rewrite this as $i\mathbf{B} \left|\psi\right\rangle = \mathbf{B} \left|\psi\right\rangle \tilde{\gamma_{12}}$ in our approach.

As we have seen we can decompose every quaternion into components with positive and negative eigenvalue under this interaction, i.e. $\mathbf{B} \left|\psi\right\rangle \tilde{\gamma_{12}} = \mathbf{B}(\left|\psi_B^+\right\rangle + \left|\psi_B^-\right\rangle)\tilde{\gamma_{12}} = |\mathbf{B}|(\left|\psi_B^+\right\rangle - \left|\psi_B^-\right\rangle)$. To interpret the meaning of this geometrially it is perhaps clearer to investigate what happens to the magnetic field under rotation by the spinor:

$$
\begin{aligned}
\left\langle\psi\right| \mathbf{B} \left|\psi\right\rangle &= -i \left\langle\psi\right| i\mathbf{B} \left|\psi\right\rangle \\
&= -i(\left\langle\psi_B^+\right| + \left\langle\psi_B^-\right|)i\mathbf{B}(\left|\psi_B^+\right\rangle + \left|\psi_B^-\right\rangle) \\
&= -i|\mathbf{B}|(\left\langle\psi_B^+\right| + \left\langle\psi_B^-\right|)(\left|\psi_B^+\right\rangle - \left|\psi_B^-\right\rangle) \\
&= -i|\mathbf{B}|(\left\langle\psi_B^+|\psi_B^+\right\rangle - \left\langle\psi_B^-|\psi_B^-\right\rangle) \\
&= \gamma_{12}|\mathbf{B}|(|\psi_B^+|^2 - |\psi_B^-|^2)P \\
&= (\gamma_{12}|\mathbf{B}||\psi_B^+|^2 - \gamma_{12}|\mathbf{B}||\psi_B^-|^2)P
\end{aligned}
$$

That is, the up component rotates the magnetic field such that it aligns with the spin axis, the down component rotates the magnetic field such that it anti-aligns. To put it another way, they are the only two options that keep the magnetic field unaffected by rotation in the spin axis.

In the SG experiment we're dealing with an inhomogeneous B-field in the z direction, so $\mathbf{B} = B_z \hat{z} \gamma_{12}$. The interaction with the spinor then becomes: $i\mathbf{B} \left|\psi\right\rangle = B_z z \left|\uparrow\right\rangle - B_z z \left|\downarrow\right\rangle$. This looks exactly identical to a potential falling or rising in the z direction, but applied to different parts of the wavefunction. Therefore the wavefunction splits into two disjoint regions in z, and once it hits the screen causing a measurement of the position to be made, the electron (or silver atom) can only appear in one of the two regions.

The Bloch sphere

The Bloch sphere is sometimes used to visualize spinors. An arrow pointing upwards represents the z-up state, an arrow pointing downwards the z-down state, other arrows on the sphere are considered complex linear combinations of these two states. The way to make sense of this is that an arrow on the Bloch sphere stands for a spinor which rotates the magnetic field from initially pointing upwards to that arrow, i.e. the arrow correponds to our earlier s direction, which we rotated γ_{12} into, and the spinor itself correponds to $R = \sqrt{-s\gamma_{12}}$. Due to the phase that a spinor has this rotation is not unique, which is sometimes visualized by a flag with various rotations on the arrow/flagpole.

Notes on relativistic spinors

Our Pauli spinor so far corresponds to either a right-handed or left-handed spinor in the Dirac theory. In making the jump to relativistic spinors there are some subtleties involved that are important to keep straight.

We consider our spinors so far to be left-handed Weyl spinors. A right-handed Weyl spinor is a mirrored version of this and as such it spins in the opposite direction. We can multiply a spinor by γ_0 to switch its chirality, so $\psi_R = \gamma_0 \psi_L$ gives us a right spinor. Since so far we have only dealt with i and quaternionic bivectors, which both commute with γ_0, the current construction does not permit us to tell a difference between a left and a right spinor. Clearly the issue is with i, which we took to refer to the *local* spin axis, which is the same for left and right spinors.

There is however another element $I = \gamma_{0123}$, which anti-commutes with all odd elements (therefore commutes with all even elements) and squares to -1. Except for its anti-commutation with odd elements it behaves identically to the complex i. This however means that it behaves with opposite sign on left and right spinors. We can therefore interpret I to be an i that can tell the difference between left and right spinors. This implies we should re-evaluate our use of i above and possibly replace it by I.

Furey takes another approach to this and puts left and right spinors into different left ideals instead, in which i has opposite meaning. TODO: yet transformations have to be conjugated for a right spinor. investigate what's going on

TODO