Spinors

We will build spinors by the method outlined in (*From Invariant Decomposition to Spinors*. 3.1. Algebraic Spinors): Find nilpotent (isotropic, null) vectors, multiply them to generate idempotents. We will use complex numbers but will see that (in our cases) they can be interpreted as just convenient notation that introduces no new geometry.

To have a notion of hermitian conjugation that is compatible with traditional approaches, we *define* it to be the combination of the negation of all negative-squaring basis vectors and the reverse.

$$\gamma_0^\dagger = \gamma_0 \qquad \sigma_i^\dagger = \sigma_i$$
$$\gamma_i^\dagger = -\gamma_i \qquad i^\dagger = -i$$
$$(ab)^\dagger = b^\dagger a^\dagger \qquad (a+b)^\dagger = a^\dagger + b^\dagger$$

Pauli Spinors

The Pauli-algebra is generated by $\sigma_1, \sigma_2, \sigma_3$. We only have a single commuting bivector: σ_{12}. We define

$$\alpha^\pm := \frac{1}{2}(\sigma_1 \pm i\sigma_2)$$

and note that

$$\begin{aligned}(\alpha^+)^2 = (\alpha^-)^2 &= 0\\ \alpha^+ + \alpha^- &= \sigma_1\end{aligned}$$

Squaring both sides of the second equation yields:

$$\begin{aligned}(\alpha^+ + \alpha^-)^2 &= \sigma_1^2\\ (\alpha^+)^2 + (\alpha^-)^2 + \alpha^+\alpha^- + \alpha^-\alpha^+ &= 1\\ \alpha^+\alpha^- + \alpha^-\alpha^+ &= 1\\ \{\alpha^+, \alpha^-\} &= 1\end{aligned}$$

Now we can show that $\alpha^+\alpha^-$ and $\alpha^-\alpha^+$ are projectors:

$$\begin{aligned}(\alpha^+\alpha^-)^2 &= \alpha^+\alpha^-\alpha^+\alpha^-\\ &= \alpha^+\alpha^-(1 - \alpha^-\alpha^+)\\ &= \alpha^+\alpha^- - \alpha^+(\alpha^-)^2\alpha^+\\ &= \alpha^+\alpha^- \qquad\qquad (\alpha^-\alpha^+ \text{ analogous})\\ &=: P\end{aligned}$$

And therefore

$$\begin{aligned}\alpha^+\alpha^+\alpha^- &= (\alpha^+)^2\alpha^- = 0\\ \alpha^-\alpha^-\alpha^+ &= (\alpha^-)^2\alpha^+ = 0\\ \alpha^-\alpha^+\alpha^- &= \alpha^-(1 - \alpha^-\alpha^+) = \alpha^-\\ \alpha^+\alpha^-\alpha^+ &= \alpha^+(1 - \alpha^+\alpha^-) = \alpha^+\end{aligned}$$

We can now make the identification

$$\begin{aligned}|{\uparrow}\rangle &\equiv 1P = \alpha^+\alpha^-\\ |{\downarrow}\rangle &\equiv \alpha^-P = \alpha^-\end{aligned}$$

The correpsonding bras are obtained by taking the hermitian conjugate:

$$\begin{aligned}
(\alpha^{\pm})^\dagger &= \alpha^{\mp} \\
P^\dagger = (\alpha^+\alpha^-)^\dagger &= \alpha^+\alpha^- = P \\
\langle\uparrow| &= P1 = \alpha^+\alpha^- \\
\langle\downarrow| &= P\alpha^+ = \alpha^+
\end{aligned}$$

The inner product now behaves as expected:

$$\begin{aligned}
\langle\uparrow\mid\uparrow\rangle &= \langle\downarrow\mid\downarrow\rangle = P \\
\langle\uparrow\mid\downarrow\rangle &= \langle\downarrow\mid\uparrow\rangle = 0 \\
|\psi\rangle &= a\,|\uparrow\rangle + b\,|\downarrow\rangle \\
\langle\phi| &= \langle\uparrow|\,c^* + \langle\downarrow|\,d^* \\
\langle\phi|\psi\rangle &= (\langle\uparrow|\,c^* + \langle\downarrow|\,d^*)(a\,|\uparrow\rangle + b\,|\downarrow\rangle) \\
&= (c^*a + d^*b)P
\end{aligned}$$

If we let $\sigma_3 = -i\sigma_{12}$ we can express the Clifford basis in terms of the spinor basis:

$$\begin{aligned}
\sigma_1 &= \alpha^+ + \alpha^- \\
i\sigma_2 &= \alpha^+ - \alpha^- \\
\sigma_3 &= -i\sigma_{12} = \\
&= -(\alpha^+ + \alpha^-)(\alpha^+ - \alpha^-) \\
&= \alpha^+\alpha^- - \alpha^-\alpha^+
\end{aligned}$$

Multiplying these with the spinor basis elements is straightforward and gives:

$$\begin{aligned}
\sigma_1\,|\uparrow\rangle &= |\downarrow\rangle \\
\sigma_1\,|\downarrow\rangle &= |\uparrow\rangle \\
\\
\sigma_2\,|\uparrow\rangle &= i\,|\downarrow\rangle \\
\sigma_2\,|\downarrow\rangle &= -i\,|\uparrow\rangle \\
\\
\sigma_3\,|\uparrow\rangle &= |\uparrow\rangle \\
\sigma_3\,|\downarrow\rangle &= -\,|\uparrow\rangle
\end{aligned}$$

If we write spinors as column vectors, we can now simply read off the matrix representation of the Clifford basis vectors.

$$\begin{aligned}
|\uparrow\rangle &= \begin{pmatrix}1\\0\end{pmatrix} \qquad |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix} \\
\sigma_1 &= \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \\
\sigma_2 &= \begin{pmatrix}0 & -i\\i & 0\end{pmatrix} \\
\sigma_3 &= \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}
\end{aligned}$$

So we see that a matrix representation of a Clifford algebra is nothing more but the representation of the action of a multivector on a spinor in terms of a spinor basis.

We can get rid of complex i in the basis coefficients. Note that P has the eigenelement $-i\sigma_{12}$:

$$\begin{aligned} -i\sigma_{12}P &= -(\alpha^+ + \alpha^-)(\alpha^+ - \alpha^-)\alpha^+\alpha^- \\ &= (\alpha^+ + \alpha^-)\alpha^- \\ &= \alpha^+\alpha^- = P \\ iP &= \sigma_{12}P \end{aligned}$$

That is, we can exchange a complex i for a σ_{12} at the projector: $iABCP = ABCiP = ABC\sigma_{12}P$.

Dirac Spinors

We're using the mostly-negative metric

$$\begin{aligned} \gamma_0^2 &= 1 \\ \gamma_i^2 &= -1 \end{aligned}$$

The two commuting bivectors γ_{03} and γ_{12} give us two pairs of null vectors:

$$\begin{aligned} \alpha_1^\pm &:= \frac{1}{2}(\gamma_0 \pm \gamma_3) \\ \alpha_2^\pm &:= \frac{1}{2}(i\gamma_1 \pm \gamma_2) \end{aligned}$$

Both pairs behave like $\alpha^\pm$ above, and in addition (because they have no vectors in common) they anticommute:

$$\alpha_1^\pm \alpha_2^\pm = -\alpha_2^\pm \alpha_1^\pm$$

Because their respective projectors are even-grade elements, they commute with other $\alpha^\pm$ and with each other:

$$\begin{aligned} P_i &:= \alpha_i^+ \alpha_i^- \\ \alpha_i^\pm P_j &= P_j \alpha_i^\pm \qquad\qquad i \neq j \\ P_1 P_2 &= P_2 P_1 =: P \end{aligned}$$

Their product is also a projector:

$$\begin{aligned} (P_1P_2)^2 &= P_1P_2P_1P_2 \\ &= P_1P_1P_2P_2 \\ &= P_1P_2 \end{aligned}$$

From this we can build a spinor basis:

$$\begin{aligned} |{\uparrow\uparrow}\rangle &= P \\ |{\downarrow\uparrow}\rangle &= \alpha_1^- P \\ |{\uparrow\downarrow}\rangle &= \alpha_2^- P \\ |{\downarrow\downarrow}\rangle &= \alpha_1^- \alpha_2^- P \end{aligned}$$

From our definition of $\alpha_i^\pm$ we can easily recover the γ-basis:

$$\begin{aligned} \gamma_0 &= \alpha_1^+ + \alpha_1^- \\ \gamma_3 &= \alpha_1^+ - \alpha_1^- \\ i\gamma_1 &= \alpha_2^+ + \alpha_2^- \\ \gamma_2 &= \alpha_2^+ - \alpha_2^- \end{aligned}$$

With this calculating the action of γ_0 on the spinor basis is straightforward:

$$\begin{aligned}\gamma_0 \left|\uparrow\uparrow\right\rangle &= \left|\downarrow\uparrow\right\rangle \\ \gamma_0 \left|\downarrow\uparrow\right\rangle &= \left|\uparrow\uparrow\right\rangle \\ \gamma_0 \left|\uparrow\downarrow\right\rangle &= \left|\downarrow\downarrow\right\rangle \\ \gamma_0 \left|\downarrow\downarrow\right\rangle &= \left|\uparrow\downarrow\right\rangle\end{aligned}$$

it "flips the left arrow".

Consider a spinor $\phi = a\left|\uparrow\uparrow\right\rangle + b\left|\downarrow\downarrow\right\rangle$. ϕ is even because $\left|\uparrow\uparrow\right\rangle$ and $\left|\downarrow\downarrow\right\rangle$ are even elements. Multiplying by a Lorentz transformation on the left will keep it even because these are part of the even subalgebra themselves.

We can now write a general spinor as $\psi = \phi_1 + \gamma_0\phi_2$.

A Lorentz transformation consists of boosts and rotations generated by γ_{i0} and γ_{ij} bivectors respectively. This is how they act on a general spinor:

$$\begin{aligned}\gamma_{ij}\psi &= \gamma_{ij}\phi_1 + \gamma_{ij}\gamma_0\phi_2 = \gamma_{ij}\phi_1 + \gamma_0\gamma_{ij}\phi_2 = \phi_1' + \gamma_0\phi_2' \\ \gamma_{i0}\psi &= \gamma_{i0}\phi_1 + \gamma_{i0}\gamma_0\phi_2 = \gamma_{i0}\phi_1 - \gamma_0\gamma_{i0}\phi_2 = \phi_1' - \gamma_0\phi_2'\end{aligned}$$

That is, they behave the same under a rotation but with opposite sign under a boost. ϕ_1 is therefore a left-handed and ϕ_2 a right-handed Weyl-spinor.

Now to investigate the inner product. For this we need a bra spinor basis. We find that

$$\begin{aligned}(\alpha_i^\pm)^\dagger &= \alpha_i^\mp \\ P_i^\dagger = (\alpha_i^+\alpha_i^-)^\dagger &= \alpha_i^+\alpha_i^- = P_i \\ P^\dagger &= P\end{aligned}$$

The bra and ket bases then are

$$\begin{aligned}\left\langle\uparrow\uparrow\right| &= P1 & \left|\uparrow\uparrow\right\rangle &= 1P \\ \left\langle\downarrow\uparrow\right| &= P\alpha_1^+ & \left|\downarrow\uparrow\right\rangle &= \alpha_1^- P \\ \left\langle\uparrow\downarrow\right| &= P\alpha_2^+ & \left|\uparrow\downarrow\right\rangle &= \alpha_2^- P \\ \left\langle\downarrow\downarrow\right| &= P\alpha_2^+\alpha_1^+ & \left|\downarrow\downarrow\right\rangle &= \alpha_1^-\alpha_2^- P\end{aligned}$$

Because a lone $\alpha_i^\pm$ between the projectors annihilates the term, only products of corresponding basis spinors are non-null. Their $\alpha_i^\pm$ combine to projectors and the product simplifies to P, i.e. $\langle i|j\rangle = \delta_{ij}P$.

The product $\psi^\dagger\psi$ is however not yet Lorentz invariant. Any Lorentz transformation can be expressed as the product of a commuting rotation and boost:

$$\psi \to \psi' = e^{aA}e^{bB}\psi$$

where A and B are bivectors generating a rotation and boost respectively with these properties:

$$\begin{aligned}&AB = BA && && \\ &A^2 = -1 && \gamma_0 A = A\gamma_0 && A^\dagger = -A \\ &B^2 = 1 && \gamma_0 B = -B\gamma_0 && B^\dagger = B\end{aligned}$$

The conjugate spinor transforms as follows:

$$\begin{aligned}\psi^\dagger \to \psi'^\dagger &= \psi^\dagger(e^{bB})^\dagger(e^{aA})^\dagger \\ &= \psi^\dagger e^{bB}e^{-aA}\end{aligned}$$

Clearly this is missing a negation of B to cancel out the transformation on ψ. So we define the Dirac adjoint:

$$\bar{\psi} = \psi^\dagger \gamma_0$$

The adjoint transforms as follows:

$$\begin{aligned}
\bar{\psi} \to \bar{\psi}' = \psi'^\dagger \gamma_0 &= \psi^\dagger (e^{bB})^\dagger (e^{aA})^\dagger \gamma_0 \\
&= \psi^\dagger e^{bB} e^{-aA} \gamma_0 \\
&= \psi^\dagger e^{bB} \gamma_0 e^{-aA} \\
&= \psi^\dagger \gamma_0 e^{-bB} e^{-aA} \\
&= \bar{\psi} e^{-bB} e^{-aA}
\end{aligned}$$

Then the transformations cancel out to give

$$\bar{\psi}' \psi' = \bar{\psi} \psi$$

If we want to get rid of the complex i we can do so just as in the case of the Pauli spinor by using the eigenelement $-i\gamma_{12}$ of P_2 to exchange a complex i for a γ_{12} at the projector: $iABCP = ABCiP = ABC\gamma_{12}P$.

Dirac equation

We define a derivative operator $\Diamond := \partial_t \gamma_0 + \partial_x \gamma_1 + \partial_y \gamma_2 + \partial_z \gamma_3$ and write down the Dirac equation like this:

$$(i\Diamond - m)\psi = 0$$

Plane wave solution

We start with plane waves of the form

$$\psi_\pm(x) = u_\pm(p) e^{\mp i x \cdot p}$$

where

$$\begin{aligned}
x &= t\gamma_0 + x\gamma_1 + y\gamma_2 + z\gamma_3 \\
p &= E\gamma_0 + p_x\gamma_1 + p_y\gamma_2 + p_z\gamma_3
\end{aligned}$$

Inserting into the equation yields:

$$\begin{aligned}
(i\Diamond - m) u_\pm(p) e^{\mp i x \cdot p} &= 0 \\
(i(\mp i p) - m) u_\pm(p) e^{\mp i x \cdot p} &= 0 \\
(\pm p - m) u_\pm(p) e^{\mp i x \cdot p} &= 0 \\
(\pm p - m) u_\pm(p) &= 0 \\
(m \mp p) u_\pm(p) &= 0
\end{aligned}$$

Rest frame

We let $p_i = 0$ (whereby $E = m$) and the equation becomes:

$$(1 \mp \gamma_0)u_\pm = 0$$

We write $u_\pm = \psi_L^\pm + \gamma_0\psi_R^\pm$:

$$\begin{aligned}
(1 \mp \gamma_0)(\psi_L^\pm + \gamma_0\psi_R^\pm) &= 0 \\
\psi_L^\pm \mp \psi_R^\pm + \gamma_0(\psi_R^\pm \mp \psi_L^\pm) &= 0 \\
\psi_L^+ &= \psi_R^+ \\
-\psi_L^- &= \psi_R^- \\
u_+ &= \phi_+ + \gamma_0\phi_+ \\
u_- &= \phi_- - \gamma_0\phi_-
\end{aligned}$$

The classic four solutions are then:

$$\begin{aligned}
\psi_1 &= (|\uparrow\uparrow\rangle + \gamma_0 |\uparrow\uparrow\rangle)e^{-iEt} \\
\psi_2 &= (|\downarrow\downarrow\rangle + \gamma_0 |\downarrow\downarrow\rangle)e^{-iEt} \\
\psi_3 &= (|\uparrow\uparrow\rangle - \gamma_0 |\uparrow\uparrow\rangle)e^{+iEt} \\
\psi_4 &= (|\downarrow\downarrow\rangle - \gamma_0 |\downarrow\downarrow\rangle)e^{+iEt}
\end{aligned}$$

Moving frame

Back to the general case:

$$(m \mp p)u_\pm(p) = 0$$

Note that $(m \mp p)(m \pm p) = 0$ because $p^2 = m^2$. So we let $u_\pm(p) = (m \pm p)u_\pm$.

$$\begin{aligned}
(m \pm p)u_\pm &= (m \pm E\gamma_0 \pm p_i\gamma_i)(\phi_\pm \pm \gamma_0\phi_\pm) \\
&= (m + E + p_i\gamma_{i0})\phi^\pm \\
&\pm \gamma_0(m + E - p_i\gamma_{i0})\phi^\pm
\end{aligned}$$

Massless case

If we let $m = 0$ we get the following:

$$\begin{aligned}
pu_\pm(p) &= 0 \\
p^2u_\pm(p) &= 0 \\
p^2 &= 0
\end{aligned}$$

and hence a lightlike 4-momentum.