Split Complex Numbers

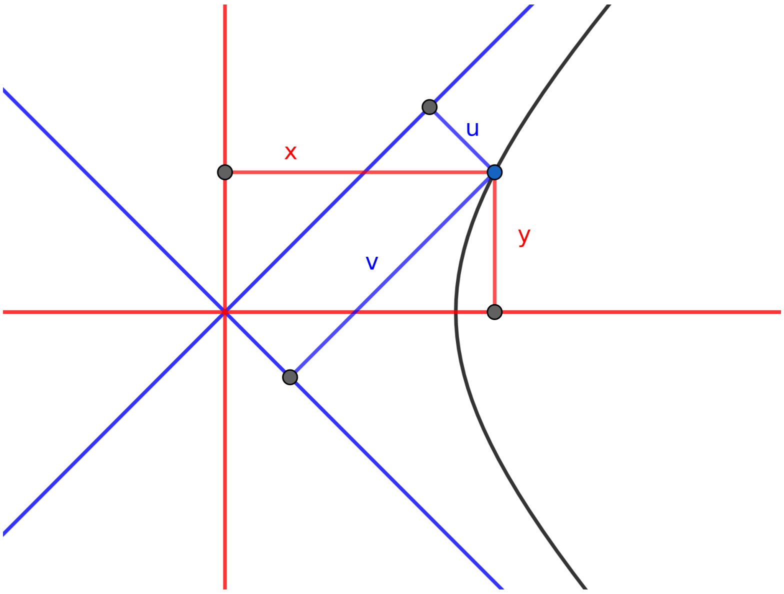
The split complex numbers are defined analogously to the regular complex numbers by introducing an element j such that $j^2 = 1$.

We then have for multiplication $(a + jb)(c + jd) = ac + bd + j(ad + bc)$ and for the squared magnitude $||a+jb||^2 := (a+jb)(a-jb) = a^2 - b^2$. This means that split complex numbers with the same magnitude lie on the same hyperbola.

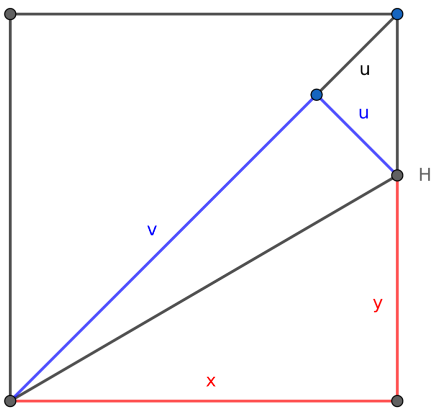
Hyperbola

There are two definitions of hyperbolas and we will first show that these are equivalent.

$$x^2 - y^2 = \text{const.}$$
$$uv = \text{const.}$$



Consider the red and blue coordinates of point H on a hyperbola.



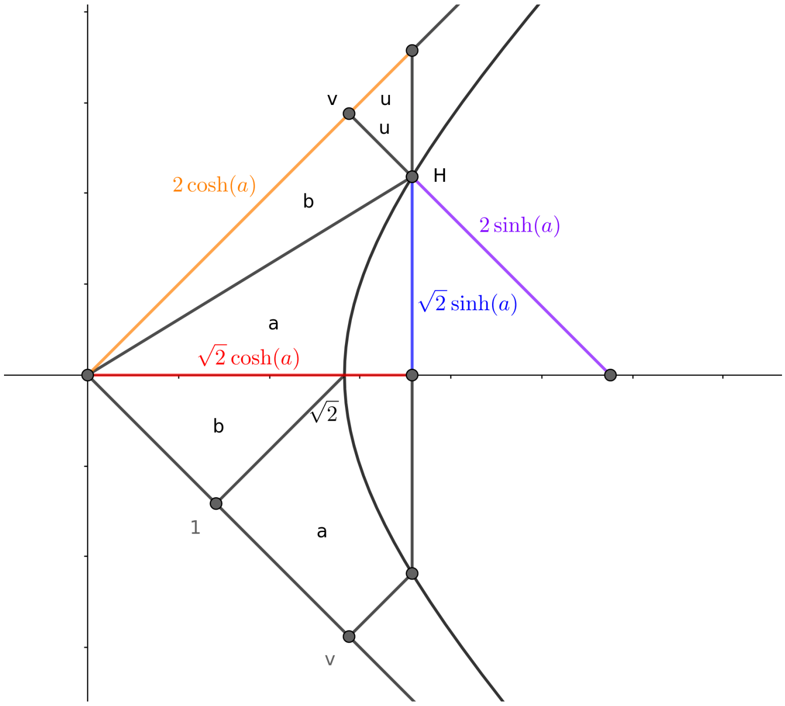
The Pythagorean theorem yields:

$$x^2 + y^2 = u^2 + v^2$$
$$x^2 + x^2 = (u + v)^2$$

and subtracting these from each other:

$$x^2 + x^2 - (x^2 + y^2) = (u + v)^2 - (u^2 + v^2)$$
$$x^2 - y^2 = 2uv$$

Since hyperbolic rotation is not as obvious to see as circular rotation we have to resort to a different way of understand split complex number multiplication. [1]



Consider the hyperbola with vertex $(\sqrt{2}, 0)$, i.e. $x^2 - y^2 = 2$ and $uv = 1$. Let H be a point on the hyperbola with coordinates $\sqrt{2}(\cosh(a), \sinh(a))$. We want to find the area a which parameterized the hyperbola by means of the hyperbolic functions $\cosh(a)$ and $\sinh(a)$.

Observe $b = \frac{1}{2}uv = \frac{1}{2}$, thus the two b's and a's resp. are indeed of equal area. The Pythagorean theorem further yields $2\cosh(a) = \sqrt{2 \cdot 2\cosh(a)^2}$ and $2\sinh(a) = \sqrt{2 \cdot 2\sinh(a)^2}$ for the orange and purple lines. We also have $2\cosh(a) = v + u$ and $2\sinh(a) = v - u$. For area a consider the bottom area:

$$a = \int_1^v \frac{1}{t} dt = \log(t)\Big|_1^v = \log(v) - \log(1) = \log(v)$$
$$\Rightarrow v = e^a$$
$$u = \frac{1}{v} = e^{-a}$$

[1] Found on stackexchange https://math.stackexchange.com/questions/757091/alternative-definition-of-hyperbolic-cosine-without-relying-on-exponential-funct/757241#757241

Thus we find

$$\begin{aligned}
\cosh(a) &= \frac{e^a + e^{-a}}{2} \\
\sinh(a) &= \frac{e^a - e^{-a}}{2} \\
e^a &= \cosh(a) + \sinh(a)
\end{aligned}$$

Note that the unit hyperbola is defined by $x^2 - y^2 = 1$ and in that case the above area a is $\frac{a}{2}$ instead.

Using the above results the hyperbolic angle addition formulas can be derived (we skip this step because it is not hard but tedious):

$$\begin{aligned}
\cosh(x+y) &= \cosh(x)\cosh(y) + \sinh(x)\sinh(y) \\
\sinh(x+y) &= \cosh(x)\sinh(y) + \sinh(x)\cosh(y)
\end{aligned}$$

Now we can finally confirm that split complex multiplication is hyperbolic rotation:

$$\begin{aligned}
&(\cosh(a) + j\sinh(a))(\cosh(b) + j\sinh(b)) \\
=&\cosh(a)\cosh(b) + \sinh(a)\sinh(b) + j(\cosh(a)\sinh(b) + \sinh(a)\cosh(b)) \\
=&\cosh(a+b) + j\sinh(a+b)
\end{aligned}$$

And it can be shown in much the same way as with complex numbers that

$$e^{jx} = \cosh(x) + j\sinh(x)$$

TODO: quadrants