Intro

This is my attempt to make sense of section 6.8 of *C. Furey, Standard model physics from an algebra?*.

$\mathbb{C}\mathbf{l}(6)$ Spinors, $\bigwedge \mathbb{C}^3$

From $\mathbb{C}\text{l}(6)$ with $e_i^2 = -1$ we can build the nilpotent objects

$$\alpha_1^\pm = \frac{1}{2}(ie_1 \pm e_4) \qquad \alpha_2^\pm = \frac{1}{2}(ie_2 \pm e_5) \qquad \alpha_3^\pm = \frac{1}{2}(ie_3 \pm e_6)$$

with the following anti-commutative property:

$$\begin{aligned} \alpha_i^+ \alpha_j^+ + \alpha_i^+ \alpha_j^+ &= 0 \\ \alpha_i^- \alpha_j^- + \alpha_i^- \alpha_j^- &= 0 \\ \Rightarrow (\alpha_i^\pm)^2 &= 0 \end{aligned}$$

Interesting to note is that the α_i^+ and α_i^- each are a basis of an exterior algebra $\bigwedge \mathbb{C}^3$ with the wedge-product just being the product. We will call these the α^+- and α^--algebras. They are related to each other by hermitian conjugation, which is defined such that it flips the nilpotent objects and reverses multiplication:

$$\begin{aligned} (\alpha_i^\pm)^\dagger &= \alpha_i^\mp \\ (ab)^\dagger &= b^\dagger a^\dagger \end{aligned}$$

We also have

$$\begin{aligned} \alpha_j^+ + \alpha_j^- &= ie_j \\ (\alpha_j^+ + \alpha_j^-)^2 &= 1 \\ \Rightarrow \alpha_j^{+-} + \alpha_j^{-+} &= 1 \end{aligned}$$

from which follows that $\alpha_i^{+-}, \alpha_i^{-+}$ are idempotent. Note also that they commute and are hermitian.

$$\begin{aligned} (\alpha_i^{-+})^2 &= \alpha_i^{-+}\alpha_i^{-+} \\ &= \alpha_i^{-+}(1 - \alpha_i^{+-}) \\ &= \alpha_i^{-+} \quad (\alpha_i^{+-}\text{analogous}) \\ \alpha_i^{-+}\alpha_j^{-+} &= \alpha_j^{-+}\alpha_i^{-+} \\ (\alpha_i^{-+})^\dagger &= \alpha_i^{-+} \end{aligned}$$

We now construct a master idempotent which we can treat as a vaccum state on which the $\alpha_i^\pm$ act as raising and lowering operators:

$$V := \alpha_1^{-+}\alpha_2^{-+}\alpha_3^{-+}$$

We will denote a general multivector in an α-algebra with lower-case ψ, ϕ. A spinor is then such a multivector left multiplied onto V, we denote these with upper-case Ψ, Φ.

ψ, Ψ and their hermitian conjugates then look like this:

$$\begin{aligned}\psi =&\psi_0\\ &+\psi_1\alpha_1^+ + \psi_2\alpha_2^+ + \psi_3\alpha_3^+\\ &+\psi_{23}\alpha_{23}^+ + \psi_{31}\alpha_{31}^+ + \psi_{12}\alpha_{12}^+\\ &+\psi_{123}\alpha_{123}^+\\ \Psi =&\psi V\\ \psi^\dagger =&\psi_0^*\\ &+\psi_1^*\alpha_1^- + \psi_2^*\alpha_2^- + \psi_3^*\alpha_3^-\\ &+\psi_{23}^*\alpha_{23}^- + \psi_{31}^*\alpha_{31}^- + \psi_{12}^*\alpha_{12}^-\\ &+\psi_{123}^*\alpha_{123}^-\\ \Psi^\dagger =&V\psi^\dagger\end{aligned}$$

Lie theory

We are interested in transformations $e^{iX}Ye^{-iX}$ where X is a generator of the Lie algebra. This can be evaluated using the Hadamard-lemma:

$$\begin{aligned}e^X Y e^{-X} &= \sum_{m=0}^{\infty} \frac{1}{m!}[X,Y]_m\\ [X,Y]_m &= [X,[X,Y]]_{m-1}\\ [X,Y]_0 &= Y\end{aligned}$$

Note this general property of the bracket (x here commutes with everything):

$$[xX,Y]_m = x^m[X,Y]_m$$

Let us consider three special cases (x again commutes with everything):

$$\begin{aligned}[X,Y] &= 0\\ \Rightarrow [X,Y]_m &= 0 \quad m>0\\ \Rightarrow e^X Y e^{-X} &= \frac{1}{0!}[X,Y]_0 = Y\end{aligned} \tag{1}$$

$$\begin{aligned}[xX,Y] &= xY\\ \Rightarrow [xX,Y]_m &= x^m Y\\ \Rightarrow e^{xX} Y e^{-xX} &= \left(\sum_{m=0}^{\infty} \frac{x^m}{m!}\right) Y = e^x Y\end{aligned} \tag{2}$$

$$\begin{aligned}[X,Y] &= XY\\ \Rightarrow [X,Y]_m &= X^m Y\\ \Rightarrow e^X Y e^{-X} &= \left(\sum_{m=0}^{\infty} \frac{X^m}{m!}\right) Y = e^X Y\end{aligned} \tag{3}$$

U(1) and SU(3) symmetries

Unitarity

We define an inner product between two spinors Φ and Ψ as $\Phi^\dagger\Psi$, which comes out to be

$$\Phi^\dagger\Psi = \sum_x \phi_x^* \psi_x V$$

where x goes over the indices of all coefficients. It is important to keep in mind that an inner product is not just a (complex) scalar but includes the master idempotent.

We wish to generate symmetries with the exponential map and require that these leave the inner product invariant. If spinors transform like this

$$\Psi \to \Psi' = e^{i\sum xX}\Psi$$
$$\Phi^\dagger \to \Phi'^\dagger = \Phi^\dagger (e^{i\sum xX})^\dagger$$

it is obvious that the condition

$$(e^{i\sum xX})^\dagger e^{i\sum xX} = 1$$
$$(e^{i\sum xX})^\dagger = e^{-i\sum xX}$$
$$e^{-i\sum xX^\dagger} = e^{-i\sum xX}$$
$$X^\dagger = X$$

will give

$$\Phi'^\dagger\Psi' = \Phi^\dagger\Psi$$

leaving the inner product invariant. That is, if the generators are hermitian then the exponential and its hermitian conjugate will be inverses of each other. In matrix formulation this is a unitary group. The reason for this unusual one-sided transformation law lies in the idempotent V as we will soon see.

Specialness

We also wish for our highest graded element of the α-algebra to stay invariant under the group action (not to pick up any phase or be negated):

$$e^{i\sum xX}\alpha_{123}^+ e^{-i\sum xX} = \alpha_{123}^+$$

This makes it a *special* unitary group.

The generators

Our symmetries should also preserve grading, i.e. we want

$$\alpha = c_1\alpha_1^+ + c_1\alpha_2^+ + c_3\alpha_3^+$$
$$e^{i\sum xX}\alpha e^{-i\sum xX} = c_1'\alpha_1^+ + c_1'\alpha_2^+ + c_3'\alpha_3^+$$

This means the generators will have to be built out of products of the same number of raising and lowering operators, which can be visualized in the following table:

	α_1^-	α_2^-	α_3^-
α_1^+	α_1^{+-}	$\alpha_1^+\alpha_2^-$	$\alpha_1^+\alpha_3^-$
α_2^+	$\alpha_2^+\alpha_1^-$	α_2^{+-}	$\alpha_2^+\alpha_3^-$
α_3^+	$\alpha_3^+\alpha_1^-$	$\alpha_3^+\alpha_2^-$	α_3^{+-}

Elements mirrored along the main diagonal are hermitian conjugates of each other. This means their sum is hermitian, and multiplied by i their difference is hermitian. The diagonal elements are already hermitian. This leaves us with 9 generators, of which the first six are these:

$$\begin{aligned}
\Lambda_1 &= \alpha_1^+\alpha_2^- + \alpha_2^+\alpha_1^- & \Lambda_2 &= i(\alpha_1^+\alpha_2^- - \alpha_2^+\alpha_1^-) \\
\Lambda_4 &= \alpha_3^+\alpha_1^- + \alpha_1^+\alpha_3^- & \Lambda_5 &= i(\alpha_3^+\alpha_1^- - \alpha_1^+\alpha_3^-) \\
\Lambda_6 &= \alpha_2^+\alpha_3^- + \alpha_3^+\alpha_2^- & \Lambda_7 &= i(\alpha_2^+\alpha_3^- - \alpha_3^+\alpha_2^-)
\end{aligned}$$

For the other three we could choose e.g. α_i^{+-}, however we can build two generators which leave α_{123}^+ invariant, and one which multiplies it by a phase factor and commutes with all the others.

The latter is the number/grade operator $N = \alpha_1^{+-} + \alpha_2^{+-} + \alpha_3^{+-}$. Because $[N, \alpha_{123}^+] = 3\alpha_{123}^+$ we have a case of (2) and therefore N generates U(1).

To get a special group recall that we need

$$e^{i\sum x_i\Lambda_i}\alpha_{123}^+ e^{-i\sum x_i\Lambda_i} = \alpha_{123}^+$$

which according to (1) is the case if $[\Lambda_i, \alpha_{123}^+] = 0$. Note that for the first six generators this is already the case because we have $\Lambda_i\alpha_{123}^+ = \alpha_{123}^+\Lambda_i = 0$. For the remaining two generators the same can be achieved by requiring that the sum of the coefficients of the projectors be zero. Finally we arrive at the full set:

$$\begin{aligned}
\Lambda_1 &= \alpha_1^+\alpha_2^- + \alpha_2^+\alpha_1^- & \Lambda_2 &= i(\alpha_1^+\alpha_2^- - \alpha_2^+\alpha_1^-) & \Lambda_3 &= \alpha_2^{+-} - \alpha_1^{+-} \\
\Lambda_4 &= \alpha_3^+\alpha_1^- + \alpha_1^+\alpha_3^- & \Lambda_5 &= i(\alpha_3^+\alpha_1^- - \alpha_1^+\alpha_3^-) & & \\
\Lambda_6 &= \alpha_2^+\alpha_3^- + \alpha_3^+\alpha_2^- & \Lambda_7 &= i(\alpha_2^+\alpha_3^- - \alpha_3^+\alpha_2^-) & & \\
\Lambda_8 &= \frac{1}{\sqrt{3}}(\alpha_1^{+-} + \alpha_2^{+-} - 2\alpha_3^{+-}) & N &= \alpha_1^{+-} + \alpha_2^{+-} + \alpha_3^{+-} & &
\end{aligned}$$

(TODO: how and why the normalization?)

This has the structure [1] of $\mathfrak{su}(3)$ with

$$\begin{gathered}
[\Lambda_j, \Lambda_k] = 2if_{jkl}\Lambda_l \\
f_{123} = 1 \\
f_{453} = f_{673} = f_{147} = f_{156} = f_{246} = -f_{157} = -\frac{1}{2} \\
f_{458} = -f_{678} = \frac{\sqrt{3}}{2}
\end{gathered}$$

and therefore the Λ_i generate SU(3).

Full-cover and Half-cover

We are now in a position to understand why spinors transform only on one side. Note that $HV = VH = 0$, where H is a linear combination of any of the 9 generators above. So we get $[H, A^+V] = HA^+V$, where A stands for any number of α_i. This is a case of (3):

$$\begin{aligned}
e^{iH}A^+Ve^{-iH} &= e^{iH}A^+V \\
\Rightarrow e^{iH}\Psi e^{-iH} &= e^{iH}\Psi
\end{aligned}$$

[1] Negate Λ_i for $i \neq 3, 5$ to get the conventional structure constants.

The conjugate spinor of course behaves analogously:

$$\begin{aligned} e^{iH} V A^- e^{-iH} &= V A^- e^{-iH} \\ \Rightarrow e^{iH} \Psi^\dagger e^{-iH} &= \Psi^\dagger e^{-iH} \end{aligned}$$

We see that our specific choice of transformations caused the half-cover sandwich to coincide with a full-cover one-sided transformation. It is important to note that not every transformation has this property, but it does suggest that a one-sided transformation for spinors is in some sense natural.

To visualize this one-sidedness in terms of the Balinese cup trick we might think of ψ as being the hand holding the cup and transforming normally under a sandwich (half-cover). The projector then can be thought of as the shoulder, which is connected to the hand/ψ. The half-cover rotation then automatically becomes a full-cover rotation.

Transformation properties

Now to investigate some transformation properties of the coefficients of the $\alpha^{\pm}$-algebras. Let $U = \sum x_i \Lambda_i$ be any SU(3) action:

$$\begin{aligned} \alpha &= c_1 \alpha_1^+ + c_1 \alpha_2^+ + c_3 \alpha_3^+ \\ \rightarrow U \alpha U^\dagger &= c_1' \alpha_1^+ + c_1' \alpha_2^+ + c_3' \alpha_3^+ \\ (U \alpha U^\dagger)^\dagger = U \alpha^\dagger U^\dagger &= c_1'^* \alpha_1^- + c_1'^* \alpha_2^- + c_3'^* \alpha_3^- \end{aligned}$$

We can see that α and $\alpha^\dagger$ transform with conjugated coefficients, that is, α transforms as a 3 and $\alpha^\dagger$ as a $\bar{3}$.

To find how grade-2 elements transform we introduce the notion of a Hodge dual. If $\alpha(\star\alpha) = \alpha_{123}^+$ then $\star\alpha$ is the Hodge dual of α.

Consider the product

$$(c_1 \alpha_1^+ + c_2 \alpha_2^+ + c_3 \alpha_3^+)(c_{23} \alpha_{23}^+ + c_{31} \alpha_{31}^+ + c_{12} \alpha_{12}^+) = (c_1 c_{23} + c_2 c_{31} + c_3 c_{12}) \alpha_{123}^+$$

Assuming α is normalized we find

$$\begin{aligned} \alpha &= c_1 \alpha_1^+ + c_2 \alpha_2^+ + c_3 \alpha_3^+ \\ \star\alpha &= c_1^* \alpha_{23}^+ + c_2^* \alpha_{31}^+ + c_3^* \alpha_{12}^+ \\ \alpha(\star\alpha) &= \alpha_{123}^+ \end{aligned}$$

Then recall $U \alpha_{123}^+ U^\dagger = \alpha_{123}^+$:

$$\begin{aligned} U \alpha_{123}^+ U^\dagger &= U \alpha (\star\alpha) U^\dagger = (U \alpha U^\dagger)(U (\star\alpha) U^\dagger) \\ &= \alpha' (\star\alpha)' = \alpha' (\star\alpha') = \alpha_{123}^+ \end{aligned}$$

That is, the transformed dual is the dual of the transformed element. If α is a grade-1 element transforming as a 3, then its grade-2 dual $\star\alpha$ has to transform as a $\bar{3}$. Similarly we find that $\star(\alpha^\dagger)$ has to transform as a 3 because $\alpha^\dagger$ transforms as a $\bar{3}$.