Triality

inner product -> polarization identity/parallelogram law.
$|x+y|^2 = |x|^2 + |y|^2 = 2 < x, y >$
element as column-vector

O(1): ± 1

derive duality symmetries (isometries): O(1), O(2), O(4), O(8)

automorphism groups: 1, O(1), SO(3), G_2. these leave the real part unchanged

Spin(2)	SO(2)	U(1)		
Spin(3)	SO(3)	SU(2)	Sp(1)	
Spin(4)	SO(4)			Spin(3) $\times$ Spin(3)
Spin(5)	SO(5)		Sp(2)(?)	
Spin(6)	SO(6)	SU(4)(?)		
Spin(7)	SO(7)			
Spin(8)	SO(8)			SO(8) $\times$ SO(8)

Complex triality

$\mathbb{C}$-triality, Cl(2,0):

$$\overline{s}\,V s = \begin{pmatrix}\overline{s}_L & \overline{s}_R\end{pmatrix}\begin{pmatrix} & \overline{v} \\ v & \end{pmatrix}\begin{pmatrix} s_L \\ s_R\end{pmatrix} = \overline{s_R}\,v s_L + \overline{s_L}\,\overline{v}\,s_R$$

where $s_R, v, s_L \in \mathbb{C}$ and $V = a\sigma_1 + b\sigma_2 \in \mathrm{Cl}^1(2,0)$. The Cl(3,0) matrix rep makes the structure clearer but no third dimension is involved.

A Spin(2) element has the form $R = UV = \begin{pmatrix}\overline{u}\,v & \\ & u\,\overline{v}\end{pmatrix} = \begin{pmatrix} z & \\ & \overline{z}\end{pmatrix}$.

The spinors and V transform under R as follows:

$$Rs = \begin{pmatrix} z & \\ & \overline{z}\end{pmatrix}\begin{pmatrix} s_L \\ s_R\end{pmatrix} = \begin{pmatrix} z s_L \\ \overline{z}\,s_R\end{pmatrix}$$

$$RVR^{-1} = \begin{pmatrix} z & \\ & \overline{z}\end{pmatrix}\begin{pmatrix} & \overline{v} \\ v & \end{pmatrix}\begin{pmatrix} \overline{z} & \\ & z\end{pmatrix} = \begin{pmatrix} & z\,\overline{v}\,z \\ \overline{z}\,v\,\overline{z} & \end{pmatrix}$$

Quaternionic triality

$\mathbb{H}$-triality, Cl(4,0):

$$\overline{s}\,V s = \begin{pmatrix}\overline{s}_L & \overline{s}_R\end{pmatrix}\begin{pmatrix} & \overline{v} \\ v & \end{pmatrix}\begin{pmatrix} s_L \\ s_R\end{pmatrix} = \overline{s_R}\,v s_L + \overline{s_L}\,\overline{v}\,s_R$$

where $s_R, v, s_L \in \mathbb{H}$ and $V \in \mathrm{Cl}^1(4,0)$.

A simple Spin(4) element has the form $R = UV = \begin{pmatrix}\overline{u}\,v & \\ & u\,\overline{v}\end{pmatrix}$. If we let u and v be purely imaginary the conjugate simplifies and we get: $R_a = -\begin{pmatrix} uv & \\ & uv\end{pmatrix} = \begin{pmatrix} q & \\ & q\end{pmatrix}$ with any $q \in \mathrm{Sp}(1)$ because every unit quaternion is the product of two purely-imaginary ones. On the other hand, now let $u = 1$ and v be any q, then we get: $R_b = \begin{pmatrix} q & \\ & \overline{q}\end{pmatrix}$.

From these we form two products, also in Spin(4):

$$\begin{aligned} R_a R_b &= \begin{pmatrix} q & \\ & q \end{pmatrix} \begin{pmatrix} q & \\ & \overline{q} \end{pmatrix} = \begin{pmatrix} q^2 & \\ & 1 \end{pmatrix} \\ R_a \widetilde{R_b} &= \begin{pmatrix} q & \\ & q \end{pmatrix} \begin{pmatrix} \overline{q} & \\ & q \end{pmatrix} = \begin{pmatrix} 1 & \\ & q^2 \end{pmatrix} \end{aligned}$$

Where q^2 is an arbitrary unit quaternion again, of which we can take the square root. Only if $q^2 = -1$ is this ambiguous, in which case the Spin element is the pseudoscalar e_{1234} that splits the space into left- and right-handed components in the first place.

We saw that any Spin(4)-element is a pair of quaternions by construction, but also that these quaternions are independent (unlike in the complex case), and that in fact any pair of unit quaternions gives a valid Spin(4) element. So we find that Spin(4) lets us rotate the left- and right-handed parts independently under Spin(3), and therefore $\mathrm{Spin}(4) \cong \mathrm{Spin}(3) \times \mathrm{Spin}(3)$.

Now to investigate how the vector and spinors transform:

$$\begin{aligned} Rs &= \begin{pmatrix} a & \\ & b \end{pmatrix} \begin{pmatrix} s_L \\ s_R \end{pmatrix} = \begin{pmatrix} as_L \\ bs_R \end{pmatrix} \\ RVR^{-1} &= \begin{pmatrix} a & \\ & b \end{pmatrix} \begin{pmatrix} & \overline{v} \\ v & \end{pmatrix} \begin{pmatrix} \overline{a} & \\ & \overline{b} \end{pmatrix} = \begin{pmatrix} & a\,\overline{v}\,\overline{b} \\ bv\,\overline{a} & \end{pmatrix} = \begin{pmatrix} & \overline{bv\,\overline{a}} \\ bv\,\overline{a} & \end{pmatrix} \end{aligned}$$

The expression $bv\bar{a}$ is well known to transform the quaternion v under SO(4), which can also be shown in other ways. (TODO: do this?)

Octonionic triality

$\mathbb{O}$-triality, Cl(8,0):

$$\overline{s}\,Vs = \begin{pmatrix} \overline{s}_L & \overline{s}_R \end{pmatrix} \begin{pmatrix} & L_{\overline{v}} \\ L_v & \end{pmatrix} \begin{pmatrix} s_L \\ s_R \end{pmatrix} = \overline{s_R}(vs_L) + \overline{s_L}(\overline{v}\,s_R)$$

where $s_R, v, s_L \in \mathbb{O}$ and $V \in \mathrm{Cl}^1(8,0)$.

A simple Spin(8) element has the form $R = PQ = \begin{pmatrix} L_{\overline{p}}L_q & \\ & L_pL_{\overline{q}} \end{pmatrix}$.

Now to investigate how the vector and spinors transform:

$$\begin{aligned} Rs &= \begin{pmatrix} L_{\overline{p}}L_q & \\ & L_pL_{\overline{q}} \end{pmatrix} \begin{pmatrix} s_L \\ s_R \end{pmatrix} = \begin{pmatrix} \overline{p}(qs_L) \\ p(\overline{q}\,s_R) \end{pmatrix} \\ RVR^{-1} &= \begin{pmatrix} L_{\overline{p}}L_q & \\ & L_pL_{\overline{q}} \end{pmatrix} \begin{pmatrix} & L_{\overline{v}} \\ L_v & \end{pmatrix} \begin{pmatrix} L_{\overline{q}}L_p & \\ & L_qL_{\overline{p}} \end{pmatrix} \end{aligned}$$